Complex Variables
Preliminary Exam
January 2021

Directions: Do all of the following eight problems. **Show all your work and justify your answers.** Each problem is worth 10 points.

Notation: $\mathbb{C}$ — the complex plane; $\mathbb{D} := \{z : |z| < 1\}$ — the unit disk; $\Re(z)$ and $\Im(z)$ denote the real part of z and the imaginary part of z, respectively.

1. **(a)** Give the definition of a complex-valued function $f(z)$ differentiable at a point $z_0 \in \mathbb{C}$.
(b) Give the definition of the derivative $f'(z_0)$ of a complex-valued function $f(z)$ at a point $z_0 \in \mathbb{C}$.
(c) Give the definition of a complex-valued function $f(z)$ analytic at a point $z_0 \in \mathbb{C}$.
(d) Let $g(z) = x^2y^2 + i\,0$. Find all points $z = x + iy \in \mathbb{C}$, where:
(1) $g(z)$ satisfies the Cauchy-Riemann equations,
(2) $g(z)$ is differentiable,
(3) $g(z)$ is analytic.

2. Prove Cauchy's Estimates for Taylor coefficients of analytic functions.

3. Let $f(z)$ be an entire function such that $f(0) = \alpha$, $\alpha \in \mathbb{C}$, and $|f(z)f(1/z)| \leq 1$ for all $z \in \mathbb{C} \setminus \{0\}$. Determine for which values of α the function $f(z)$ has
(a) removable singularity at ∞;
(b) pole at ∞;
(c) essential singularity at ∞.

4. Use the Residue Calculus to evaluate the integral

$$\int_0^\infty \frac{1+x^2}{1+x^4}\,dx$$

5. Let $a \in \mathbb{C}$ and let $n \in \mathbb{N}$, $n \geq 2$. Prove that the equation

$$1 + z + az^n = 0$$

has at least one solution in the disk $\{z : |z| \leq 2\}$.

(**Hint:** For $|a|$ large enough, you may invoke Vieta's formula for the product of roots of a complex polynomial.)

6. Let $\Omega \subset \mathbb{C}$ be a domain, $z_0 \in \Omega$, and let $\mathcal{F}(\Omega) = \{f : \Omega \to \mathbb{D} : f \text{ is analytic in } \Omega\}$. Suppose that $g \in \mathcal{F}(\Omega)$ and

$$|g'(z_0)| = \sup_{f \in \mathcal{F}(\Omega)} |f'(z_0)|.$$

Prove that $g(z_0) \cdot g'(z_0) = 0$.

7. Locate and classify all singularities (including a possible singularity at $z = \infty$) of:

(a) $e^{\frac{1}{z}} - \dfrac{1}{z} + \dfrac{1}{e^z - 1}$ **(b)** $\dfrac{z-1}{\operatorname{Log} z}$ **(c)** $z\tan(1/z)$

8. For $n \in \mathbb{N}$, let

$$f_n(z) = \frac{1}{1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \cdots + \frac{(-1)^n}{(2n)!}z^{2n}}$$

and let R_n be the radius of convergence of the power series expansion of f_n about $z = 0$.

Find $\lim_{n\to\infty} R_n$.