Directions: Do all of the following eight problems. Show all your work and justify your answers. Each problem is worth 10 points.

Notation: \( \mathbb{C} \) — the complex plane; \( z = x + iy \in \mathbb{C} \); \( \mathbb{D} := \{ z : |z| < 1 \} \) — the unit disk; \( \Re(z) \) and \( \Im(z) \) denote the real part of \( z \) and the imaginary part of \( z \), respectively.

1. (a) Give the definition of a function \( f(z) \) analytic in the unit disk \( \mathbb{D} \).
   
   (b) Give the definition of a function \( f(z) \) analytic at the point \( z = 0 \).
   
   (c) Let \( f(z) = u(x,y) + iv(x,y) \) be analytic on \( \mathbb{C} \) such that \( u(x,y)v(x,y) = 1 \) for all \( z = x + iy \in \mathbb{D} \). Prove that \( f(z) \) is constant on \( \mathbb{C} \).

2. (a) Find the Taylor expansion at \( z = 0 \) of the function \( k(z) = z^{1/2} - z^{2/3} \).
   
   (b) Use the Cauchy-Hadamard Formula to find the radius of convergence of the Taylor series of \( k(z) \) in part (a).

3. Use the Residue Theory to evaluate the following integrals:
   
   (a) \( \int_0^\infty \frac{\cos(2x)}{4 + x^2} \, dx \) 
   
   (b) \( \int_0^{2\pi} \frac{\sin \theta}{\sqrt{5 + \sin \theta}} \, d\theta \).

4. Find all entire functions \( f(z) \) for which \( f(0) = f'(0) = 0 \) and for each \( n \in \mathbb{N} \), \( \max_{|z|=n} |f(z)| = n^2 \).

5. (a) State the Argument Principle.
   
   (b) Find the number of solutions of the equation \( e^z = z^2 + 50 \) in the rectangle \( \{ z = x + iy : 0 < x < 3, |y| < 4 \} \).

6. Let \( f(z) \) be a function that is analytic and satisfies \( |f(z)| \leq 1 \) in the unit disk \( \mathbb{D} \). Prove that if \( f(z) \) has at least two distinct fixed points in \( \mathbb{D} \) then \( f(z) = z \) for all \( z \in \mathbb{D} \).

7. Find a conformal mapping \( f : \Omega \to \mathbb{H} \) from the domain \( \Omega = \{ z : |\arg z| < \pi/3 \} \setminus [0,1] \) onto the upper half-plane \( \mathbb{H} = \{ z : \Im(z) > 0 \} \) such that \( f(2) = 1 + i \).

8. (a) State the Weierstrass Product Theorem.
   
   (b) Give an example of an entire function \( f(z) \) whose only zeros are simple zeros at the points \( z_n = in, n \in \mathbb{Z} \).
   
   (c) Give an example of a meromorphic on \( \mathbb{C} \) function \( g(z) \) whose only poles are simple poles at the points \( z_n = n, n \in \mathbb{N} \), and whose only zeros are simple zeros at the points \( \zeta_n = -n, n \in \mathbb{N} \).
   
   (d) Classify the singularity at \( z = \infty \) for each of the functions \( f(z) \) and \( g(z) \) from parts (b) and (c).