Complex Variables
Preliminary Exam
May 2021

Directions: Do all of the following eight problems. **Show all your work and justify your answers.** Each problem is worth 10 points.

Notation: $\mathbb{C}$ — the complex plane; $z = x + iy \in \mathbb{C}$; $\mathbb{D} := \{z : |z| < 1\}$ — the unit disk; $\Re(z)$ and $\Im(z)$ denote the real part of z and the imaginary part of z, respectively.

1. **(a)** Give the definition of a function $f(z)$ analytic in the unit disk $\mathbb{D}$.
(b) Give the definition of a function $f(z)$ analytic at the point $z = 0$.
(c) Let $f(z) = u(x, y) + iv(x, y)$ be analytic on $\mathbb{C}$ such that $u(x, y)v(x, y) = 1$ for all $z = x + iy \in \mathbb{D}$. Prove that $f(z)$ is constant on $\mathbb{C}$.

2. **(a)** Find the Taylor expansion at $z = 0$ of the function $k(z) = \frac{z}{1-z^2}$.
(b) Use the Cauchy-Hadamard Formula to find the radius of convergence of the Taylor series of $k(z)$ in part **(a)**.

3. Use the Residue Theory to evaluate the following integrals:

$$(\mathbf{a}) \quad \int_0^\infty \frac{\cos(2x)}{4 + x^2}\, dx \qquad\qquad (\mathbf{b}) \quad \int_0^{2\pi} \frac{\sin\theta}{\sqrt{5} + \sin\theta}\, d\theta.$$

4. Find all entire functions $f(z)$ for which $f(0) = f'(0) = 0$ and for each $n \in \mathbb{N}$, $\max_{|z|=n} |f(z)| = n^2$.

5. **(a)** State the Argument Principle.
(b) Find the number of solutions of the equation $e^z = z^2 + 50$ in the rectangle $\{z = x + iy : 0 < x < 3, |y| < 4\}$.

6. Let $f(z)$ be a function that is analytic and satisfies $|f(z)| \le 1$ in the unit disk $\mathbb{D}$. Prove that if $f(z)$ has at least two distinct fixed points in $\mathbb{D}$ then $f(z) = z$ for all $z \in \mathbb{D}$.

7. Find a conformal mapping $f : \Omega \to \mathbb{H}$ from the domain $\Omega = \{z : |\arg z| < \pi/3\} \setminus [0, 1]$ onto the upper half-plane $\mathbb{H} = \{z : \Im(z) > 0\}$ such that $f(2) = 1 + i$.

8. **(a)** State the Weierstrass Product Theorem.
(b) Give an example of an entire function $f(z)$ whose only zeros are simple zeros at the points $z_n = in$, $n \in \mathbb{Z}$.
(c) Give an example of a meromorphic on $\mathbb{C}$ function $g(z)$ whose only poles are simple poles at the points $z_n = n$, $n \in \mathbb{N}$, and whose only zeros are simple zeros at the points $\zeta_n = -n$, $n \in \mathbb{N}$.
(d) Classify the singularity at $z = \infty$ for each of the functions $f(z)$ and $g(z)$ from parts **(b)** and **(c)**.