Complex Variables
Preliminary Exam
August 2022

Directions: Do all of the following eight problems. **Show all your work and justify your answers.** Each problem is worth 10 points.

Notation: $\mathbb{C}$ — the complex plane; $z = x + iy \in \mathbb{C}$; $D(z, r) = \{w \in \mathbb{C} : |w - z| < r\}$ — the open disk centered at $z \in \mathbb{C}$ and having radius $r > 0$; $\mathbb{D} = \{z : |z| < 1\}$ — the unit disk; $\Re(z)$ and $\Im(z)$ denote the real part of z and the imaginary part of z, respectively.

1. Solve the following problems:

(a) Let $f(z) = |z|^2$, $z \in \mathbb{C}$. Find the points where f has a complex derivative and the points where f is holomorphic.

(b) Find the singularities (including a possible singularity at ∞) of the function

$$f(z) = e^{1/z} + \frac{1}{2 - z - z^2}.$$

Classify each singularity as removable, pole or essential.

2. Let

$$f(z) = \frac{1}{z^2 + 1}.$$

Find the Taylor series of f centered at $z = 1$ and the radius of convergence of this series.

3. State and prove the argument principle.

4. Solve the following problems:

(a) Let γ be a differentiable closed curve in $\mathbb{C}$ such that $0 \in \mathbb{C} \setminus \gamma$ and $Ind_\gamma(0) = -1$, where $Ind_\gamma(0)$ denotes the index (winding number) of γ about $z = 0$. Find

$$\int_\gamma \frac{e^z(z^2 - 1)}{z} dz.$$

(b) Find the principal value of the integral

$$I = \int_{-\infty}^{+\infty} \frac{\cos(x)}{x^2 + 1} dx.$$

5. Let f be an entire function satisfying $|f(z)| \le C|z|^n$, for all $z \in \mathbb{C}$ with $|z| > 100$, for some $n \in \mathbb{N}$ and some $C > 0$. Prove that f is a polynomial of degree at most n.

6. Find the number of zeros of $p(z) = z^4 + 6z - 1$ in the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

7. Let $\mathcal{F}$ be the family of all holomorphic functions $f : \mathbb{D} \to \mathbb{D}$.

(a) State Montel's theorem and use it to show that there exists a function $F \in \mathcal{F}$ that maximizes $|f'(1/2)|$, over all $f \in \mathcal{F}$. In other words, show that

$$\sup_{f \in \mathcal{F}} \left| f'\left(\frac{1}{2}\right) \right| = \left| F'\left(\frac{1}{2}\right) \right|,$$

for some $F \in \mathcal{F}$.

(b) Use Schwarz's lemma to determine all extremal functions F from part **(a)**.

8. Find a linear fractional transformation mapping the domain $D = \mathbb{D} \setminus \overline{D(1/4, 1/4)}$ onto an annulus $\mathbb{D} \setminus \overline{D(0, r)}$ centered at the origin, for some $r \in (0, 1)$.