Complex Variables
Preliminary Exam
August 2022

Directions: Do all of the following eight problems. Show all your work and justify your answers. Each problem is worth 10 points.

Notation: \( \mathbb{C} \) — the complex plane; \( z = x + iy \in \mathbb{C} \); \( D(0,r) = \{ w \in \mathbb{C} : |w-z| < r \} \) — the open disk centered at \( z \in \mathbb{C} \) and having radius \( r > 0 \); \( \mathbb{D} = \{ z : |z| < 1 \} \) — the unit disk; \( \Re(z) \) and \( \Im(z) \) denote the real part of \( z \) and the imaginary part of \( z \), respectively.

1. Solve the following problems:
   (a) Let \( f(z) = |z|^2 \), \( z \in \mathbb{C} \). Find the points where \( f \) has a complex derivative and the points where \( f \) is holomorphic.
   (b) Find the singularities (including a possible singularity at \( \infty \)) of the function
   \[
   f(z) = e^{1/z} + \frac{1}{2-z-z^2}.
   \]
   Classify each singularity as removable, pole or essential.

2. Let
   \[
   f(z) = \frac{1}{z^2 + 1}.
   \]
   Find the Taylor series of \( f \) centered at \( z = 1 \) and the radius of convergence of this series.

3. State and prove the argument principle.

4. Solve the following problems:
   (a) Let \( \gamma \) be a differentiable closed curve in \( \mathbb{C} \) such that \( 0 \in \mathbb{C} \setminus \gamma \) and \( \text{Ind}_\gamma(0) = -1 \), where \( \text{Ind}_\gamma(0) \) denotes the index (winding number) of \( \gamma \) about \( z = 0 \). Find
   \[
   \int_\gamma \frac{e^{z^2-1}}{z} \, dz.
   \]
   (b) Find the principal value of the integral
   \[
   I = \int_{-\infty}^{+\infty} \frac{\cos(x)}{x^2 + 1} \, dx.
   \]

5. Let \( f \) be an entire function satisfying \( |f(z)| \leq C|z|^n \), for all \( z \in \mathbb{C} \) with \( |z| > 100 \), for some \( n \in \mathbb{N} \) and some \( C > 0 \). Prove that \( f \) is a polynomial of degree at most \( n \).

6. Find the number of zeros of \( p(z) = z^4 + 6z - 1 \) in the annulus \( A = \{ z \in \mathbb{C} : 1 < |z| < 2 \} \).

7. Let \( \mathcal{F} \) be the family of all holomorphic functions \( f : \mathbb{D} \to \mathbb{D} \).
   (a) State Montel’s theorem and use it to show that there exists a function \( F \in \mathcal{F} \) that maximizes \( |f'(1/2)| \), over all \( f \in \mathcal{F} \). In other words, show that
   \[
   \sup_{f \in \mathcal{F}} |f'\left(\frac{1}{2}\right)| = |F'\left(\frac{1}{2}\right)|,
   \]
   for some \( F \in \mathcal{F} \).
   (b) Use Schwarz’s lemma to determine all extremal functions \( F \) from part (a).

8. Find a linear fractional transformation mapping the domain \( D = \mathbb{D} \setminus \overline{D}(1/4,1/4) \) onto an annulus \( \mathbb{D} \setminus \overline{D}(0, r) \) centered at the origin, for some \( r \in (0, 1) \).