Complex Variables
Preliminary Exam
May 2022

Directions: Do all of the following eight problems. **Show all your work and justify your answers.** Each problem is worth 10 points.

Notation: $\mathbb{C}$ — the complex plane; $z = x + iy \in \mathbb{C}$; $D(z, r) = \{w \in \mathbb{C} : |w - z| < r\}$ — the open disk centered at $z \in \mathbb{C}$ and having radius $r > 0$; $\mathbb{D} = \{z : |z| < 1\}$ — the unit disk; $\Re(z)$ and $\Im(z)$ denote the real part of z and the imaginary part of z, respectively.

1. Prove the following:

(a) Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function in $\mathbb{C}$. Prove that

$$\frac{\partial f}{\partial x}(z) = -i\frac{\partial f}{\partial y}(z), \qquad \text{for every } z = x + iy \in \mathbb{C}.$$

(b) Suppose that f is holomorphic in the disk $D(a, r)$, $a \in \mathbb{C}$, $r > 0$, and $\Im(f)$ is constant in $D(a, r)$. Prove that f is constant in $D(a, r)$.

2. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. Prove that

$$\left|\frac{f(z) - f(w)}{1 - \overline{f(w)}f(z)}\right| \leq \left|\frac{z - w}{1 - \overline{w}z}\right|, \qquad z, w \in \mathbb{D},$$

and

$$|f'(w)| \leq \frac{1 - |f(w)|^2}{1 - |w|^2}, \qquad w \in \mathbb{D}.$$

3. State and prove Liouville's theorem.

4. Use the residue theorem to compute the following definite integral:

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos(2\theta)}{1 + 2\cos^2(\theta)} d\theta.$$

5. Let n and m be two positive integers.

(a) Prove that for every $z \in \mathbb{C}$ with $|z| \leq 1$,

$$\left|1 + z + \frac{z^2}{2} + \cdots + \frac{z^m}{m!}\right| \leq e.$$

(b) Prove that the polynomial

$$p(z) = 1 + z + \frac{z^2}{2} + \cdots + \frac{z^m}{m!} + 3z^n,$$

has exactly n zeros in the unit disk $\mathbb{D}$, counting the orders of the zeros.

6. Do the following:

(a) State the identity principle for holomorphic and for harmonic functions.

(b) Suppose that f is an entire function satisfying $f(n) = n$, for $n \in \mathbb{N}$, and

$$\lim_{z \to \infty} |f(z)| = +\infty.$$

Show that $f(z) = z$, for every $z \in \mathbb{C}$.

7. Prove that there exists a sequence of polynomials p_n that converges pointwise in the complex plane $\mathbb{C}$ to the function f defined by

$$f(z) = \begin{cases} 1, & \Im(z) > 0, \\ 0, & \Im(z) = 0, \\ -1, & \Im(z) < 0. \end{cases}$$

8. Do the following:

(a) State the Riemann mapping theorem.

(b) Construct a conformal mapping from the domain

$$D = \left\{ z \in \mathbb{C} \setminus \{0\} : |\mathrm{Arg}(z)| < \frac{\pi}{4} \right\} \setminus [0, 1]$$

onto the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$.