Complex Variables
Preliminary Exam
August 2023

Directions: Do all of the following problems. **Show all your work and justify your answers.**

Notation: $\mathbb{C}$ — the complex plane; $\mathbb{D} := \{z : |z| < 1\}$ — the unit disk; $x = \Re(z)$ and $y = \Im(z)$ denote the real part of z and the imaginary part of z, respectively; $H(G) = \{f : G \to \mathbb{C} \, : \, f \text{ is analytic } \}$.

1. State and prove Rouche's Theorem.

2. Find all possible values of $\displaystyle\int_\gamma \frac{e^z}{(z+1)(z-4)}\, dz$ for all closed rectifiable curves γ in $\mathbb{C} \setminus \{-1, 4\}$. You may give your answer in terms of winding numbers.

3. Compute the Laurent series for $f(z) = \dfrac{z+9}{z^2+3z}$ which converges uniformly on
(a) $\{z \, : \, 1 \leq |z| \leq 2\}$
(b) $\{z \, : \, 3 \leq |z-2| \leq 4\}$

4. Suppose $f \in H(\mathbb{C})$ and $\displaystyle\int_{-\pi}^{\pi} \Re\left(f'(a+e^{it}) \right) dt > 0$ for all $a \in \mathbb{C}$. Prove that f is one-to-one.

5. For each of the following, give an example or prove that no such example exists.
a. A Möbius transformation with exactly 3 distinct fixed points.
b. A harmonic function on a region G which does not have a harmonic conjugate on G.

6. Find a conformal map f from the plane with 2 slits $\mathbb{C} \setminus \left((-\infty, -5] \cup [5, \infty) \right)$ onto the unit disc $\mathbb{D}$ such that $f(1) = 0$.

7. Consider the polynomials $p_n(z) = 1 + 2z + \frac{2^2}{2!}z^2 + \cdots + \frac{(2)^n}{n!}z^n$ and let

$$r_n = \min\{|a| \, : \, a \text{ is a zero of } p_n\}.$$

Show that $r_n \to \infty$ as $n \to \infty$.

8. Suppose G is simply connected and $a \in G$. Further supose $g : G \to G$ is analytic with $g(a) = a$.
a. If $G \neq \mathbb{C}$, prove $|g'(a)| \leq 1$.
b. If $G = \mathbb{C}$, prove $|g'(a)|$ can take on any value in $\mathbb{R}$.