Directions: Do all of the following problems. Show all your work and justify your answers.

Notation: $\mathbb{C}$ — the complex plane; $\mathbb{D} := \{z : |z| < 1\}$ — the unit disk; $x = \Re(z)$ and $y = \Im(z)$ denote the real part of $z$ and the imaginary part of $z$, respectively; $H(G) = \{f : G \to \mathbb{C} : f \text{ is analytic}\}$.

1. State and prove Rouche’s Theorem.

2. Find all possible values of $\int_{\gamma} \frac{e^z}{(z+1)(z-4)} \, dz$ for all closed rectifiable curves $\gamma$ in $\mathbb{C} \setminus \{-1, 4\}$. You may give your answer in terms of winding numbers.

3. Compute the Laurent series for $f(z) = \frac{z + 9}{z^2 + 3z}$ which converges uniformly on
   (a) $\{z : 1 \leq |z| \leq 2\}$
   (b) $\{z : 3 \leq |z-2| \leq 4\}$

4. Suppose $f \in H(\mathbb{C})$ and $\int_{-\pi}^{\pi} \Re\left( f'(a + e^{it}) \right) \, dt > 0$ for all $a \in \mathbb{C}$. Prove that $f$ is one-to-one.

5. For each of the following, give an example or prove that no such example exists.
   a. A M"obius transformation with exactly 3 distinct fixed points.
   b. A harmonic function on a region $G$ which does not have a harmonic conjugate on $G$.

6. Find a conformal map $f$ from the plane with 2 slits $\mathbb{C} \setminus ((-\infty,-5] \cup [5,\infty))$ onto the unit disc $\mathbb{D}$ such that $f(1) = 0$.

7. Consider the polynomials $p_n(z) = 1 + 2z + \frac{z^2}{2!}z^2 + \cdots + \frac{(2)^n}{n!}z^n$ and let $r_n = \min\{|a| : a \text{ is a zero of } p_n\}$.
   Show that $r_n \to \infty$ as $n \to \infty$.

8. Suppose $G$ is simply connected and $a \in G$. Further suppose $g : G \to G$ is analytic with $g(a) = a$.
   a. If $G \neq \mathbb{C}$, prove $|g'(a)| \leq 1$.
   b. If $G = \mathbb{C}$, prove $|g'(a)|$ can take on any value in $\mathbb{R}$.