**Complex Variables**
**Preliminary Exam**
May 2023

**Directions:** Do all of the following problems. **Show all your work and justify your answers.**

**Notation:** $\mathbb{C}$ — the complex plane; $\mathbb{D} := \{z : |z| < 1\}$ — the unit disk; $x = \Re(z)$ and $y = \Im(z)$ denote the real part of $z$ and the imaginary part of $z$, respectively; $H(G) = \{f : G \to \mathbb{C} : f \text{ is analytic }\}$.

**1.** State and prove Schwarz's Lemma.

**2.** Find a conformal map from the slit half-disc $\{x + iy \in \mathbb{D} : y > 0\} \setminus \{iy : 0 \le y \le \frac{1}{2}\}$ onto the strip $\{x + iy : 0 < y < 1\}$.

**3.** Evaluate $\displaystyle\int_0^\infty \frac{x^{2/3}}{(x+3)(x^2+4)}\,dx$ using the Residue Theorem.

**4.** Suppose $0 < |a_n| < |a_{n-1}| < \cdots < |a_0|$. Prove that $na_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ has no zeros in $\mathbb{D}$.

**5.** Suppose $f \in H(\mathbb{C})$ and $f'$ is $M$-Lipschitz; that is, for all $z, w \in \mathbb{C}$, we have the inequality $|f'(z) - f'(w)| \le M|z - w|$. Prove that $f$ is a polynomial of degree at most 2.

**6.** Suppose $f$ is meromorphic on $\mathbb{C} \cup \{\infty\}$.
a. Prove that $f$ has at most finitely many poles.
b. Use part a to prove that $f$ must be a rational function.
b. Prove that if $f$ is one-to-one, then $f$ must be a Möbius transformation.

**7.** If $f$ is analytic on $\mathbb{C}$ and $\Re f(x + iy) = e^x \sin(y)$, find $\Im f(x + iy)$.

**8.** Let $G$ denote the right-half plane $\{x + iy : x > 0\}$. A $G$-automorphism is a one-to-one analytic function which maps $G$ onto itself. Describe all $G$-automorphisms. That is, what form must a $G$-automorphism have? Why?