Complex Variables
Preliminary Exam
August 2024

Directions: Do all of the following eight problems. **Show all your work and justify your answers.** Each problem is worth 10 points.

Notation: $\mathbb{C}$ — the complex plane; $\mathbb{D} := \{z : |z| < 1\}$ — the unit disk; $z = x + iy$, $x = \Re(z)$ and $y = \Im(z)$ denote the real part of z and the imaginary part of z, respectively.

1. Describe and then graph the following subsets of the complex plane: **(a)** $\{z : |z+2i| - |z-2i| = 2\}$, **(b)** $\{z : \Im(z(1-i)) < \sqrt{2}\}$, **(c)** $\{z : \frac{\pi}{4} < \arg(z+i) < \frac{\pi}{2}, |z+i| > 1\}$.
2. State and then prove the theorem on the Cauchy's Estimates.
3. Find all $k \in \mathbb{Z}$ for which there exists an analytic function $f_k(z)$ on the disk $D_2 := \{z : |z| < 2\}$ such that
$$f_k\left(\frac{1}{n}\right) = f_k\left(-\frac{1}{n}\right) = \frac{1}{n^k}$$
for all positive integers n.
4. Use the Residue Calculus to evaluate the following integrals:
$$\text{(a)} \quad \int_0^{2\pi} \frac{d\theta}{4 + \sin\theta} \qquad \text{(b)} \quad \int_0^{\infty} \frac{x^2}{1 + x^{10}}\, dx.$$
5. **(a)** Determine how many poles, counting multiplicity, the rational function
$$R(z) = \frac{1 - z^2}{z^5 - z^4 + 4z^3 + z}$$
has in the unit disk $\mathbb{D}$.
(b) Let $p_n(z) = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots + \frac{1}{n!}z^n$, $n \in \mathbb{N}$. Prove that there is $N \in \mathbb{N}$ such that if $n \geq N$, then $p_n(z)$ does not have zeros in the unit disk $\mathbb{D}$.
6. Suppose $f : \mathbb{D} \to \mathbb{C}$ is a complex valued function such that $g = f^2$ and $h = f^3$ are both analytic on $\mathbb{D}$ and $|h(z)| < |g(z)|$ for all $z \in \mathbb{D}$. Prove that f is analytic on $\mathbb{D}$ and $|f'(0)| < 1$.
7. Find a conformal mapping $w = f(z)$ from the half-disk $\mathbb{D}_+ := \{z : |z| < 1, \Im(z) > 0\}$ onto the half-plane $\mathbb{H}_r = \{w : \Re(w) > 0\}$ such that $\Re(f(i/2)) = 1$ and $\arg(f(i/2)) = -\pi/3$.
8. **(a)** State the Weierstrass Factorization Theorem for an entire function.
(b) Construct an entire function that has second order zeros at the points $z_n = in$, $n \in \mathbb{Z} \setminus \{0\}$, simple zero at $z = 0$ and no other zeros.