**Complex Variables**
**Preliminary Exam**
May 2024

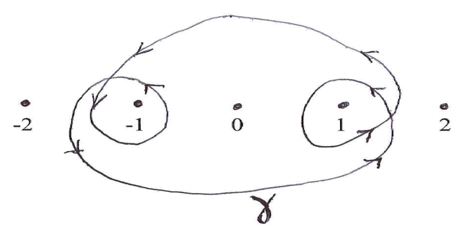
**Directions:** Do all of the following eight problems. **Show all your work and justify your answers.** Each problem is worth 10 points.

**Notation:** $\mathbb{C}$ — the complex plane; $\mathbb{D} := \{z : |z| < 1\}$ — the unit disk; $z = x + iy$, $x = \Re(z)$ and $y = \Im(z)$ denote the real part of $z$ and the imaginary part of $z$, respectively.

**1.** **(a)** Let $f : D \to \mathbb{C}$ be a complex-valued function defined on a domain $D \subset \mathbb{C}$. State the criterion of analyticity of $f(z)$ on $D$ in terms of the Cauchy-Riemann equations.
**(b)** Prove that the real and imaginary parts of the function $f(z) = \sqrt{|xy|}$ satisfy the Cauchy-Riemann equations at $z = 0$.
**(c)** Determine whether or not $f(z) = \sqrt{|xy|}$ is differentiable at $z = 0$.

**2.** Let
$$f(z) = \frac{\pi}{\sinh(\pi z)} + e^{1/z^2} + \frac{2z}{1+z^2}.$$
Locate and classify all the singularities of $f(z)$ (including any singularity at $z = \infty$) as isolated or non-isolated. Further, classify the isolated singularities by type (removable, pole, essential). Calculate the residues of $f(z)$ at its poles.

**3.** Let $f(z)$ be a function analytic in the strip $St = \{z : |\Re(z)| < 2\}$ such that $f(0) = 2$ and $f(-1) - f(1) = 4$. Evaluate the integral
$$\int_\gamma \frac{f(z)\sin(\pi z/2)}{e^{2\pi i z} - 1}\, dz,$$
where the contour $\gamma \subset St$ is shown in the figure below.




**4.** Let $f(z)$ be a non-constant entire function such that $|f(z)| \le |ze^z|$ for all $z$ such that $|z| > 100$. Prove that $f(z)$ has an essential singularity at $\infty$.

**5.** **(a)** State the Argument Principle for a function $f(z)$ analytic on a domain $D$.
**(b)** Let $f(z) = z^{100} - 8z^{10} + 3z^5 + z^2 - z - 1$. How many zeros (counting multiplicity) does $f(z)$ have in the closed unit disk $\overline{\mathbb{D}}$?

**6.** **(a)** Find a conformal mapping $\varphi(z)$ from the unit disk $\mathbb{D}$ onto the half-plane $\mathbb{H} = \{w : \Re(w) > 0\}$ such that $\Im\varphi(0) = 0$ and $\varphi'(0) = i$.
**(b)** Is there a conformal mapping $\psi(z)$ from the unit disk $\mathbb{D}$ onto itself such that $\psi(0) = -1/2$, $\psi(1/2) = 0$, and $\psi(-1/2) = 1/2$?

**7.** **(a)** State (any version of) Runge's theorem.
**(b)** Prove that there is a sequence of polynomials $p_n(z)$, $n = 1, 2, \ldots$, which converges to $\sin(\pi z)$ on the disk $\{z : |z+2| \le 1\}$ but does not converge to $\sin(\pi z)$ at any point of the disk $\{z : |z-2| \le 1\}$.

**8.** Let $f : \mathbb{D} \to \mathbb{D}$ be analytic on the unit disk $\mathbb{D}$ such that $f(0) = 0$. Prove that
$$\Im e^{if(z)} \le \sqrt{e} \quad \text{for all } z \text{ such that } |z| \le 1/2.$$