Complex Variables
Preliminary Exam
May 2025

Directions: Do all of the following problems. **Show all your work and justify your answers.**

Notation: $\mathbb{C}$ denotes the complex plane; $\mathbb{R}$ denotes the real line; $\mathbb{Z}$ denotes the integers; $B(a;r) = \{z \in \mathbb{C} : |z-a| < r\}$; $\mathbb{D} := \{z : |z| < 1\} = B(0;1)$ denotes the unit disk; $\operatorname{ann}(a,r,R) = \{z \in \mathbb{C} : r < |z-a| < R\}$.

1. Suppose $u : \mathbb{C} \to \mathbb{R}$ and $v : \mathbb{C} \to \mathbb{R}$ satisfy the Cauchy-Riemann equations.
   a. Prove that $f(z) = \overline{u(\bar{z}) + iv(\bar{z})}$ is analytic.
   b. If u is bounded, must it be true that v is also bounded? Provide a proof or give a counterexample.

2. Does there exist a meromorphic function on $\mathbb{C}$ with poles of order 3 exactly at the points $\log(n)$ and zeros of order 2 exactly at the points $\sqrt[3]{n}\, i$, for $n = 1, 2, 3, \ldots$? Either give an example of such a function, or prove such a function cannot exist.

3. a. Describe the branches of $z^{1/3}$ defined of $\mathbb{C} \setminus (-\infty, 0]$. How many branches are there? Why? What is the domain and range of each? Find $f(i)$ for each branch f.
   b. Describe the branches of $\log z$ defined of $\mathbb{C} \setminus [0, \infty)$. How many branches are there? Why? What is the domain and range of each? Find $f(i)$ for each branch f.

4. Find all possible values of $\displaystyle\int_\gamma \frac{e^{3z}}{z^2\,(z-1)}\, dz$, for all possible closed, rectifiable curves γ in $\mathbb{C} \setminus \mathbb{Z}$.

5. State and prove the Open Mapping Theorem.

6. a. Prove there exists a sequence of polynomials that converges uniformly on all compact subsets of $\mathbb{C} \setminus (-\infty, 0]$ to $\dfrac{1}{z}$.
   b. Prove there does not exist a sequence of polynomials that converges uniformly on all compact subsets of $\operatorname{ann}(1;2;3)$ to $\dfrac{1}{z}$.

7. Use Rouche's Theorem to prove the Fundamental Theorem of Algebra. That is, given a polynomial p of degree n, prove that there is ball $B(0;R)$ of radius R centered at 0 so that p has n zeros inside the ball and no zeros outside.

8. Prove there exists a unique value $r > 0$ so that there exists a unique conformal map f from the strip $S = \{x + iy : -1 < y < 1\}$ onto $B(0;r)$ with $f(0) = 0$ and $f'(0) = 1$. Find both r and f.