

**Mathematical Finance Preliminary Examination**  
**May 2025**

**Instruction:** Solve three (3) of the four (4) problems provided. Specify in the table below which problem solutions you want graded. You will be graded on solution clarity and detail.

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**Problem 1.** (*Utility Maximization in a Binomial Tree*)

Consider a financial market consisting of a stock  $\mathcal{S}$  and a riskless bank account  $\mathcal{B}$ . Time  $t = 0, 1, \dots, N$  is discrete. The price of the stock follows a binomial model

$$S_{t+1} = \begin{cases} S_t u & \text{with probability } p, \\ S_t d & \text{with probability } 1 - p, \end{cases}$$

where  $0 < p < 1$ . The value of the bank account evolves as

$$B_t = (1 + r)^t,$$

where  $r$  is the riskless interest rate. To ensure no-arbitrage in the market, we require  $0 < d < 1 + r < u$ .

Let  $\mathbb{P}$  denote the natural (historical) probability measure and  $\tilde{\mathbb{P}}$  denote the risk-neutral measure. The Radon-Nikodym derivative is

$$Z := \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}.$$

An investor, with initial wealth  $P_0 > 0$ , invests in  $\mathcal{S}$  and  $\mathcal{B}$  via a self-financing portfolio  $P_t$ ,

$$P_t = \Delta_t S_t + B_t.$$

The investor is free to choose  $\Delta_t$ , the number of shares of  $\mathcal{S}$  held at time  $t$ , with the remaining wealth invested in  $\mathcal{B}$ . The investor's wealth evolves as:

$$P_{t+1} = \Delta_t S_{t+1} + (1 + r)(P_t - \Delta_t S_t), \quad t = 0, 1, \dots, N - 1. \quad (1)$$

Consider the utility function  $U(x) = \ln(x)$ . The investor wants to maximize  $\mathbb{E}[U(P_N)]$  (subject to the self-funding constraint (1) and initial wealth  $P_0$ ).

Derive the maximizing terminal wealth  $P_N^*$  and show

$$P_N^* = \frac{P_0(1 + r)^N}{Z}.$$

**Problem 2.** (*Hedging with zero coupon bonds in the binomial model*)

Consider the binomial pricing model  $(\Omega, \mathbb{F} = \{\mathcal{F}_n, n = 0, 1, \dots, N\}, \mathbb{P})$  where  $\Omega = \{(w_1, \dots, w_N)\}$  with

$$w_n = \begin{cases} 1 & \text{(success),} \\ 0 & \text{(failure),} \end{cases} \quad n = 1, \dots, N.$$

The model has filtration  $\mathbb{F}$ , natural probability measure  $\mathbb{P}$  and equivalent risk-neutral measure  $\tilde{\mathbb{P}}$ . Let  $S_n, n = 0, 1, \dots, N$ , be the price of a risky asset (stock) having the dynamics

$$S_{n+1} = \begin{cases} S_n u & \text{w. p. } p, \\ S_n d & \text{w. p. } 1 - p, \end{cases} \quad n = 0, \dots, N - 1, \quad S_0 > 0,$$

where  $0 < p < 1$  and  $0 < d < u$ .

Let  $R_0, R_1, \dots, R_{N-1}$  be an  $\mathbb{F}$ -adapted interest rate process with  $d < R_n < u$ ,  $\mathbb{P}$ -a.s. for  $n = 0, 1, \dots, N - 1$ . Define the discount factor:

$$D_n = \frac{1}{(1 + R_0)(1 + R_1) \dots (1 + R_{n-1})}, \quad n = 1, \dots, N, \quad D_0 = 1.$$

Let  $B_{n,m} = \tilde{\mathbb{E}}_n \left( \frac{D_m}{D_n} \right)$  be the time- $n$  price of a zero-coupon bond maturing at time  $m > n$  ( $m \leq N$ ). At time  $n < m$ , the forward price for a time- $m$  delivery of the stock (the  $m$ -forward price at time  $n$ ) is

$$\text{For}_{n,m} = \frac{S_n}{B_{n,m}}$$

and the  $m$ -futures price is

$$\text{Fut}_{n,m} = \tilde{\mathbb{E}}_n[S_m].$$

(i) Assume a trader enters a long position in a forward contract at time  $n$  for delivery at time  $m > n$ .

- (a) Show that the value at time  $n$  of this position is zero.
- (b) Derive the value of the forward contract (the position) at time  $n + 1$ .

(ii) Suppose that  $R_n = r$  (a constant),  $n = 0, 1, \dots, N - 1$ . At each time  $n < m$ , the long position consists of  $(1 + r)^{m-n-1}$  such forward contracts maturing at  $m$ . Show that the total payoff from these contracts is

$$\text{Fut}_{n+1,m} - \text{Fut}_{n,m}.$$

Confirm this using the risk-neutral pricing formula.

**Problem 3.** (Markov property from quadratic variation in symmetric random walk)

Let  $M_0, M_1, M_2, \dots$  denote a symmetric random walk with  $M_0 = 0, M_{n+1} = M_n + X_{n+1}, n = 0, 1, 2, \dots$ , where  $X_n \in \{-1, +1\}$  are i.i.d. binary random variables with  $\mathbb{P}[X_n = +1] = \mathbb{P}[X_n = -1] = 1/2$ . Consider the process

$$I_n = \sum_{j=0}^{n-1} M_j (M_{j+1} - M_j), \quad n = 1, 2, \dots, \quad I_0 = 0.$$

(i) Show that

$$I_n = \frac{1}{2} (M_n^2 - n).$$

(ii) Let  $f(\cdot)$  be any real-valued function. Let  $g(\cdot)$  be defined by the conditional expectation

$$g(I_n) = \mathbb{E}[f(I_{n+1}) | \mathcal{F}_n] = \mathbb{E}_n[f(I_{n+1})], \quad n = 1, 2, \dots.$$

- (a) For every such function  $f(\cdot)$ , show that the function  $g(\cdot)$  exists and derive an explicit expression for it.  
(b) Conclude that  $(I_n)_{n \geq 0}$  is a Markov process.

**Problem 4.** (Itô's formula and SDE for transformed geometric Brownian motion)

Let  $S_t$  be a geometric Brownian motion given by the stochastic differential equation (SDE)

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where  $\mu \in \mathbb{R}, \sigma > 0$  and  $W_t$  is a standard Brownian motion. Let  $Y_t = g(S_t, t)$  for some twice continuously differentiable function  $g: \mathbb{R}^+ \times [0, T] \rightarrow \mathbb{R}$ .

(i) State Itô's formula for the function  $g(S_t, t)$ .

(ii) Let  $g(x) = x^p$ , with  $p > 0$ . Let  $Y_t = S_t^p$ . Use Itô's formula to derive the SDE satisfied by  $Y_t$ .