Mathematical Finance Preliminary Examination

May 2026

Instructions: Solve three (3) of the four (4) problems provided. Specify in the table below which problem solutions you want graded. You will be graded on solution clarity, mathematical correctness, and detail.

Problem 1. (Replication and pricing in the binomial model)

Consider a one-stock financial market in discrete time $n = 0,1,2$. The stock price process is given by the binomial model

$$S_{n+1} = \begin{cases} uS_n, & \text{with probability } p, \\ dS_n, & \text{with probability } 1-p, \end{cases}$$

where $0 < d < 1 + r < u,\ r > -1,$ and $S_0 > 0$. The bank account satisfies

$$B_n = (1+r)^n, \qquad n = 0,1,2.$$

Let $K > 0$ and consider the European call option with maturity $n = 2$ and payoff

$$X = (S_2 - K)^+.$$

(i) Compute the risk-neutral probability $\tilde{p}$.

(ii) Write down the time $n = 2$ option values at the four terminal nodes.

(iii) Compute the time $n = 1$ option values at the up-node and down-node.

(iv) Derive the time $n = 0$ price V_0 of the option.

(v) At time $n = 0$ find the replicating portfolio (Δ_0, β_0), where Δ_0 is the number of shares of stock held and β_0 is the amount invested in the bank account.

Problem 2. (Martingale property and stopping in symmetric random walk)

Let $\{X_n\}_{n\geq 1}$ be i.i.d. random variables with

$$\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = \frac{1}{2}.$$

Define the symmetric random walk

$$M_0 = 0, \qquad M_n = \sum_{k=1}^{n} X_k\,, \qquad n \geq 1.$$

(i) Show that $\{M_n\}_{n\geq 0}$ is a martingale with respect to its natural filtration $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$.

(ii) Show that the process

$$N_n := M_n^2 - n, \qquad n \geq 0,$$

is also a martingale.

(iii) Let

$$\tau = \inf\{n \geq 0 : M_n = a \text{ or } M_n = -b\},$$

where a, b are positive integers. Assuming that optional stopping may be applied to the martingale M_n, show that

$$\mathbb{P}(M_\tau = a) = \frac{b}{a+b}, \qquad \mathbb{P}(M_\tau = -b) = \frac{a}{a+b}.$$

(iv) Assuming that optional stopping may be applied to the martingale N_n, show that

$$\mathbb{E}[\tau] = ab.$$

Problem 3. (Self-financing portfolios and logarithmic utility in a one-period model)

Consider a one-period market with times 0 and 1. The stock price satisfies

$$S_1 = \begin{cases} uS_0, & \text{with probability } p, \\ dS_0, & \text{with probability } 1-p, \end{cases}$$

where $0 < d < 1 + r < u$, $0 < p < 1$, and $S_0 > 0$. The bank account evolves as

$$B_1 = (1+r)B_0, \qquad B_0 = 1.$$

An investor starts with initial wealth $x > 0$ and chooses Δ shares of stock at time 0. The remaining wealth is invested in the bank account. Thus, the terminal wealth is

$$X_1 = \Delta S_1 + (1+r)(x - \Delta S_0).$$

The investor has logarithmic utility

$$U(y) = \ln(y), \qquad y > 0,$$

and wishes to maximize

$$\mathbb{E}[\ln(X_1)].$$

(i) Express X_1 in the up-state and down-state.

(ii) Show that the optimization problem is equivalent to maximizing

$$f(\Delta) = p\ln((1+r)x + \Delta S_0(u-1-r)) + (1-p)\ln((1+r)x + \Delta S_0(d-1-r)).$$

(iii) Find the first-order condition for the optimal choice Δ^*.

(iv) Solve explicitly for Δ^*.

(v) State the condition under which the optimal terminal wealth is strictly positive in both states.

Problem 4. (Itô's formula and Black–Scholes asset pricing)

Let $\{W_t\}_{t\geq 0}$ be a standard Brownian motion on a filtered probability space, and let the stock price satisfy the geometric Brownian motion SDE

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \qquad S_0 > 0,$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants.

(i) Let $Y_t = \ln \, S_t$. Use Itô's formula to derive the SDE satisfied by Y_t and hence obtain an explicit formula for S_t.

(ii) Let $V = V(t, S), \; V \in C^{1,2}([0, T] \times (0, \infty))$, be the price of a derivative security written on S_t. State Itô's formula for $V(t, S_t)$.

(iii) Consider a self-financing portfolio that is long one derivative $V(t, S_t)$ and short Δ_t shares of stock. Show that by choosing

$$\Delta_t = V_S(t, S_t)$$

the diffusion term is eliminated.

(iv) Assuming the resulting portfolio must earn the risk-free rate r, derive the Black–Scholes partial differential equation

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0.$$

(v) For a European claim with payoff $V(T, S) = h(S)$, write down the terminal condition for the PDE.