Mathematical Finance Preliminary Examination

May 2023

Instruction: Please solve 3 out of the 4 problems provided below and specify in the designated box which 3 problems you want to be graded. Make sure to provide clear and detailed solutions.

Problem 1. Consider a stock $S$ with one-share price process $S_t, t = 0, \ldots, N$. The present time is $t = 0$, and $S_0 > 0$. The riskless bank account $B$ has with value $B_t = (1 + r)^t, t = 0, \ldots, N$, where $r$ is the riskless short rate. The price process $S_t, t = 0, \ldots, N$ follows $N$-period binomial pricing tree

$$S_{t+1} = \begin{cases} S_t u, & \text{with probability } p, \\ S_t d, & \text{with probability } q = 1 - p, \end{cases}$$

where $p \in (0,1)$, and $0 < d < 1 + r < u$. Thus, the arithmetic returns,

$$R_{t+1} = \frac{S_{t+1} - S_t}{S_t} = \begin{cases} u, & \text{with probability } p, \\ d, & \text{with probability } q = 1 - p, \end{cases}, \quad t = 0, \ldots, N - 1, \quad R_0 = 0,$$

are assumed independent identically distributed random variables, determining the stochastic basis $(\Omega, \mathcal{F} = \{ \mathcal{F}_t = \sigma(R_0, \ldots, R_t), t = 0, \ldots, N\}, \mathbb{P})$,

$$\omega = (\omega_1, \ldots, \omega_N) \in \Omega, \quad \omega_i = \begin{cases} 1 \text{ (up)}, \\ 0 \text{ (down)}, \end{cases}$$

filtration $\mathcal{F}$, and natural probability measure $\mathbb{P}$. The risk-neutral measure $\mathbb{P}$ with Radon-Nikodym derivative $Z(\omega) = \frac{\mathbb{P}(\omega)}{\mathbb{P}(\omega)}$, is determined by the risk-neutral probabilities

$$\bar{p} = \frac{1 + r - d}{u - d}, \quad \bar{q} = 1 - \bar{p} = \frac{u - 1 - r}{u - d}.$$

An investor with initial wealth of $W_0 > 0$ at time $t = 0$, invests in $S$ and $B$ in a self-financing portfolio $P_t, t = 0, \ldots, N$, $P_0 = W_0$. At $t = 0$, the investor buys $\Delta_0$ shares and deposits the rest of his wealth in $B$. The self-financing portfolio dynamics, $P_t, t = 0, \ldots, N$, which represents the investor wealth process $W_t, t = 0, \ldots, N$, is given by

$$P_{t+1} = W_{t+1} = \Delta_t S_{t+1} + (1 + r)(P_t - \Delta_t S_t), \quad t = 0, \ldots, N - 1, \quad P_0 = W_0.$$

The investor’s goal is to find an $\mathcal{F}$- adapted sequence of stock allocations $\Delta_0, \ldots, \Delta_{N-1}$ that maximizes $\mathbb{E} \ln(W_N)$.
Show that the optimal allocations $\Delta_0^{(opt)}, \ldots, \Delta_{N-1}^{(opt)}$ maximizing $\mathbb{E}U(W_N)$, with $U(x) = \ln(x)$, $x > 0$, is determined by the optimal portfolio

$$W_{t+1}^{(opt)} = P_{t+1}^{(opt)} = \Delta_t^{(opt)} S_{t+1} + (1 + r) \left( P_t^{(opt)} - \Delta_t^{(opt)} S_t \right), \quad t = 0, \ldots, N - 1,$$

where:

$$P_t^{(opt)} = \frac{W_0}{\zeta_t}, \quad t = 0, 1, \ldots, N;$$

$\zeta_t$, $t = 0, 1, \ldots, N$ is the state price density process $\zeta_t = Z_t (1 + r)^{-t}$, $t = 0, \ldots, N$; and $Z_t = \mathbb{E}_n(Z) = \mathbb{E}(Z|\mathcal{F}_t)$ is the Radon-Nikodym derivative process.

**Problem 2.** Consider the binomial pricing model, $(\Omega, \mathcal{F} = \{\mathcal{F}_n, n = 0, \ldots, N\}, \mathbb{P})$,

$$\omega = (\omega_1, \ldots, \omega_N) \in \Omega, \quad \omega_1 = \begin{cases} 1 & \text{(up)} \\ 0 & \text{(down)} \end{cases},$$

filtration $\mathcal{F}$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$, natural probability measure $\mathbb{P}$, and equivalent risk-neutral measure $\mathbb{P}$. Let $S_n$, $n = 0, \ldots, N$, $S_0 > 0$ be the price of a risky asset (stock) with price dynamics determined by the pricing tree

$$S_{t+1} = \begin{cases} S_t u, & \text{with probability } p, \\ S_t d, & \text{with probability } q = 1 - p, \end{cases}$$

where $p \in (0,1)$ and $0 < d < u$. Define the interest rate process $\mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_{N-1}$ which is $\mathcal{F}$-adapted such that $d < \mathcal{R}_i < u$, $\mathbb{P}$- almost surely, $i = 0, \ldots, N - 1$. Define the discount process,

$$D_n = \frac{1}{(1 + \mathcal{R}_0)(1 + \mathcal{R}_1) \ldots (1 + \mathcal{R}_{n-1})}, \quad n = 1, \ldots, N, \quad D_0 = 1.$$

The price at time $n$, $n = 0, \ldots, m < N$ of a zero-coupon bond maturing at time $m$ is $\mathcal{B}_{n,m} = \mathbb{E}_n \left( \frac{D_m}{D_n} \right)$, where $\mathbb{E}_n(\cdot) = \mathbb{E}(\cdot | \mathcal{F}_n)$ is the conditional expectation with respect to $\mathbb{P}$. For $n = 0, \ldots, m$, the forward (delivery) price is $\text{For}_{n,m} = \frac{S_n}{\mathcal{B}_{n,m}}$ and the futures price is $\text{Fut}_{n,m} = \mathbb{E}_n(S_m)$.

(i) Suppose at each time $n = 0, \ldots, N - 1$, a trader takes a long position in the forward contract with maturity $m$, $n < m \leq N$, and sells the contract at $n + 1$. Show that this strategy generates the cash amount $\left( S_{n+1} - S_n \frac{\mathcal{B}_{n+1,m}}{\mathcal{B}_{n,m}} \right)$ at $n + 1$.

(ii) Assume the interest rate is constant, $\mathcal{R}_i = r$, $i = 0, \ldots, N - 1$. At each time $n$, $0 \leq n < m < N - 1$, the trader takes a long position of $(1 + r)^{m-n-1}$ forward contracts with maturity $m$, and sells these contracts at time $n + 1$. Show that the resulting cash flow is the same as the difference in the futures prices $\text{Fut}_{n+1,m} - \text{Fut}_{n,m}$, $0 \leq n \leq m < N - 1$. 


**Problem 3.** Let $B_t$, $t \in [0,T]$, be a standard Brownian motion generating a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$, with canonical filtration $\mathcal{F} = \{\mathcal{F}_t = \sigma(B_u, 0 \leq u \leq t), \ t \in [0,T]\}$. Let $t_{k,n} = \frac{kT}{n}$, $k = 0, ..., n$. For $\alpha \in (0,1]$ set $\tau_{k,n} = \alpha t_{k,n} + (1-\alpha)t_{k+1,n}$, $k = 0, ..., n-1$. Define the Stratonovich $\alpha$-variation of $B_t$, $t \in [0,T]$ as

$$\mathbb{S}_T^{(\alpha)} = \lim_{n \to \infty} \mathbb{S}_{T,n}^{(\alpha)}, \quad \mathbb{S}_{T,n}^{(\alpha)} = \sum_{k=0}^{n-1} \left( B\left( \tau_{k,n}^{(\alpha)} \right) - B\left( t_{k,n} \right) \right)^2.$$ 

(i) Show that $\mathbb{E}(\mathbb{S}_T^{(\alpha)}) = (1-\alpha)T$ and that the variance of $\mathbb{S}_{T,n}^{(\alpha)}$ converges to zero as $n \uparrow \infty$.

(ii) Define the Stratonovich $\alpha$-integral of $B(t), t \in [0,T]$ as

$$\int_0^T B(t)^{(\alpha)} dB(t) = \lim_{n \to \infty} \sum_{k=0}^{n-1} B\left( \tau_{k,n}^{(\alpha)} \right) \left( B\left( t_{k+1,n} \right) - B\left( t_{k,n} \right) \right).$$

Show that

$$\int_0^T B(t)^{(\alpha)} dB(t) = \frac{1}{2}B(T)^2 + \left( \frac{1}{2} - \alpha \right) T.$$ 

**Problem 4.** Let $B(t)$, $t \geq 0$, be a standard Brownian motion generating a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$, with canonical filtration $\mathcal{F} = \{\mathcal{F}_t = \sigma(B(u), 0 \leq u \leq t), \ t \geq 0\}$. Let $S_t$, $t \geq 0$, $S_0 > 0$, be the price process of a risky asset (stock) with continuous diffusion dynamics determined by the SDE,

$$dS_t = \mu_t S_t dt + \sigma_t S_t dB(t), \ t \geq 0,$$

where $\mu_t$, $t \geq 0$, and $\sigma_t$, $t \geq 0$, are $\mathcal{F}$-adapted processes satisfying the usual regularity conditions guaranteeing that the SDE for $S_t$, $t \geq 0$, has a unique strong solution. Let $\beta_t$, $t \geq 0$, be the value of a riskless bank account

$$\beta_t = e^{\int_0^t r_s \, ds}, \quad t \geq 0,$$

where $r_t > 0$, $t \geq 0$, is the instantaneous riskless (short) rate, which is $\mathcal{F}$-adapted and $\mathbb{P}\left( \max \left\{ r_t + \frac{1}{r_t}; t \geq 0 \right\} < \infty \right) = 1$. Define the market price of risk,

$$\theta_t = \frac{\mu_t - r_t}{\sigma_t}$$

and the state price density process

3
\[ \zeta_t = \exp \left\{ -\int_0^t \theta_s dB_s - \int_0^t \left( r_s + \frac{1}{2} \theta_s^2 \right) ds \right\}, \quad t \geq 0 \]

(i) Show that \( \frac{d\zeta_t}{\zeta_t} = -\theta_t dB(t) - r_t dt \).

(ii) Let \( P_t, \ t \geq 0 \), be the value of an investor’s self-financing portfolio in the stock and in the riskless bank account when he uses a portfolio process \( \Delta(t) \). Assume \( P_t \) satisfies the dynamics

\[ dP_t = \Delta(t) dS_t + r_t (P_t - \Delta(t) S_t) dt. \]

Show that \( \zeta_t P_t, \ t \geq 0 \) is an \( \mathcal{F} \)-martingale.

(iii) Let \( T \) be the investment terminal time of an investor with initial capital (wealth) \( W_0 \) at \( t = 0 \). The investor applies the self-financing strategy \( P_t, \ t \geq 0 \) in (ii) with the goal of achieving a terminal wealth \( W_T = P_T \). Show that his initial capital should be \( W_0 = P_0 = \mathbb{E}[\zeta_T W_T] \).