Mathematical Finance Preliminary Examination

May 2024

Instruction: Please solve 4 out of the 5 problems provided below and specify in the designated box which 4 problems you want to be graded. Make sure to provide clear and detailed solutions.

Problem 1. The stock price is currently \$100. It is known that at the end of two months it will be either \$110 or \$90. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a two-month European call option with a strike price of \$95? Use no-arbitrage arguments and risk-neutral valuation.

Problem 2

Consider an N-period binomial model with a utility function

$$U(x) = \frac{1}{p} x^p,$$

where $p < 1$ and $p \neq 0$. Show that the optimal wealth at time N is given by

$$X_N = \frac{X_0 (1+r)^N Z^{\frac{1}{p-1}}}{E\left[Z^{\frac{p}{p-1}}\right]},$$

where Z is the Radon- Nikodým derivative of $\tilde{P}$ with respect to P.

Problem 3. Recall the valuation of American derivative securities whose intrinsic value is permitted to be path-dependent. We consider an N-period binomial model, with up factor u, down factor d, and interest rate r, satisfying the no-arbitrage condition $0 < d < 1 + r < u$. In such a model, we define $\mathbb{S}_n$ to be the set of all stopping times that take values in the set $\{n, n+1, \dots, N, \infty\}$. In particular, the set $\mathbb{S}_0$ contains all possible stopping times. A stopping time in $\mathbb{S}_N$ can take the value N on some paths, the value ∞ on others, and can take no other value. For each $n = 0,1, \dots, N$, let G_n be a random variable depending on the first n coin tosses. An American derivative security with intrinsic value process G_n is a contract that can be exercised at any time prior to and including time N and, if exercised at time n, pays off G_n. The price process V_n for this contract is determined by the American risk-neutral pricing formula

$$V_n = \max_{\tau \in \mathbb{S}_n} \tilde{E}_n \left[\mathbb{I}_{\{\tau \le N\}} \frac{G_\tau}{(1+r)^{\tau-n}} \right], \qquad n = 0,1,\dots,N. \qquad (1)$$

Consider the following modification of (1). Again, for each n, where $n = 0,1,\dots,N$, let G_n be a random variable depending on the first n coin tosses. The time-zero value of a derivative security that can be exercised at any time $n \le N$ for payoff G_n , but **must be exercised at time N if it has not been exercised before that time,** is

$$V_0 = \max_{\tau \in \mathbb{S}_0, \tau \le N} \tilde{E} \left[\frac{G_\tau}{(1+r)^{\tau-n}} \right], \qquad \tilde{E} = \tilde{E}_0 .$$

In contrast to formula (1), here we consider only stopping times that take one of the values $0,1,\dots,N$ and not the value ∞. Consider $G_n = K - S_n$, the derivative that permits its owner to sell one share of stock for payment K at any time up to and including N, but if the owner does not sell by time N, then she must do so at time N. Show that the optimal exercise policy is to sell the stock at time zero and that the value of this derivative is $K - S_0$.

Problem 4. Assume that a fair coin is tossed repeatedly. The probability of head on each toss is ½ , as is the probability of tail. Let $X_j = 1$ if the j'th toss results in a head and $X_j = -1$ if the j'th toss results in a tail. Consider the symmetric random walk $M_0, M_1, M_2, \dots$ defined by $M_0 = 0$ and

$$M_n = X_1 + \cdots + X_n, \qquad n \ge 1. \qquad (3)$$

Define $I_0 = 0$ and

$$I_n = \sum_{j=0}^{n-1} M_j (M_{j+1} - M_j), n = 1,2,\dots .$$

(i) Show that

$$I_n = \frac{1}{2} M_n^2 - \frac{n}{2}.$$

(ii) Let n be an arbitrary nonnegative integer and let $f(i)$ be an arbitrary function of a variable i. In terms of n and f, define another function $g(i)$ satisfying

$$E_n[f(I_{n+1})] = g(I_n). \qquad (4)$$

Note that, although the function $g(I_n)$ on the right-hand side of (4) may depend on n, the only random variable that may appear in its argument is I_n ; the random variable M_n may not appear. Show that $I_0, I_1, I_2, \ldots$ is a Markov process. (You will need to use the formula in part (i).)

Problem 5. Suppose that $G_t = g(S_t, t), t \geq 0$, where $g(x, t),\ x > 0,\ t \geq 0$, is a sufficiently smooth real-valued function, and $S_t,\ t \geq 0$, is the stock price, which follows geometric Brownian motion,

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \qquad t \geq 0, \quad S_0 > 0.$$

(i) Derive the stochastic differential equation (SDE) for G_t, where $B_t,\ t \geq 0$, is a standard Brownian motion.

(ii) Consider the process $Z_t = S_t^{\delta},\ t \geq 0$, where $\delta \neq 0$ is a real constant. Find the SDE for $Z_t, t \geq 0$.