Mathematical Finance Preliminary Examination
May 2025

Instruction: Solve three (3) of the four (4) problems provided. Specify in the table below which problem solutions you want graded. You will be graded on solution clarity and detail.

Problem 1. *(Utility Maximization in a Binomial Tree)*

Consider a financial market consisting of a stock $\mathcal{S}$ and a riskless bank account and a bank account $\mathcal{B}$. Time $t = 0, 1, \dots, N$ is discrete. The price of the stock follows a binomial model

$$S_{t+1} = \begin{cases} S_t u & \text{with probability} \quad p, \\ S_t d & \text{with probability} \quad 1-p, \end{cases}$$

where $0 < p < 1$. The value of the bank account evolves as

$$B_t = (1+r)^t \,,$$

where r is the riskless interest rate. To ensure no-arbitrage in the market, we require $0 < d < 1 + r < u$.

Let $\mathbb{P}$ denote the natural (historical) probability measure and $\widetilde{\mathbb{P}}$ denote the risk-neutral measure. The Radon-Nikodym derivative is

$$Z := \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}.$$

An investor, with initial wealth $P_0 > 0$, invests in $\mathcal{S}$ and $\mathcal{B}$ via a self-financing portfolio P_t,

$$P_t = \Delta_t S_t + B_t.$$

The investor is free to choose Δ_t, the number of shares of $\mathcal{S}$ held at time t, with the remaining wealth invested in $\mathcal{B}$. The investor's wealth evolves as:

$$P_{t+1} = \Delta_t S_{t+1} + (1+r)(P_t - \Delta_t S_t), \qquad t = 0, 1, \dots, N-1. \tag{1}$$

Consider the utility function $U(x) = \ln(x)$. The investor wants to maximize $\mathbb{E}[U(P_N)]$ (subject to the self-funding constraint (1) and initial wealth P_0).

Derive the maximizing terminal wealth P_N^* and show

$$P_N^* = \frac{P_0 Z}{(1+r)^N}\,.$$

Problem 2. *(Hedging with zero coupon bonds in the binomial model)*

Consider the binomial pricing model $(\Omega, \mathbb{F} = \{\mathcal{F}_n, n = 0, 1, \dots, N\}, \mathbb{P})$ where $\Omega = \{(w_1, \dots, w_N)\}$ with

$$w_n = \begin{cases} 1 \text{ (success)}, \\ 0 \text{ (failure)}, \end{cases} \quad n = 1, \dots, N.$$

The model has filtration $\mathbb{F}$, natural probability measure $\mathbb{P}$ and equivalent risk-neutral measure $\widetilde{\mathbb{P}}$. Let S_n, $n = 0, 1, \dots, N$, be the price of a risky asset (stock) having the dynamics

$$S_{n+1} = \begin{cases} S_n u & \text{w.p.} \quad p, \\ S_n d & \text{w.p.} \quad 1-p, \end{cases} \quad n = 0, \dots, N-1, \;\; S_0 > 0,$$

where $0 < p < 1$ and $0 < d < u$.
Let $R_0, R_1, \dots, R_{N-1}$ be an $\mathbb{F}$-adapted interest rate process with $d < R_n < u$, $\mathbb{P}$-a.s. for $n = 0, 1, \dots, N-1$. Define the discount factor:

$$D_n = \frac{1}{(1+R_0)(1+R_1)\dots(1+R_{n-1})}, \qquad n = 1, \dots, N, \qquad D_0 = 1.$$

Let $B_{n,m} = \widetilde{\mathbb{E}}_n \left({D_m}/{D_n} \right)$ be the time-n price of a zero-coupon bond maturing at time $m > n$ $(m \le N)$. At time $n < m$, the forward price for a time-m delivery of the stock (the m-forward price at time n) is

$$\mathrm{For}_{n,m} = \frac{S_n}{B_{n,m}}$$

and the m-futures price is

$$\mathrm{Fut}_{n,m} = \widetilde{\mathbb{E}}_n[S_m].$$

(i) Assume a trader enters a long position in a forward contract at time n for delivery at time $m > n$.

(a) Show that the value at time n of this position is zero.
(b) Derive the value of the forward contract (the position) at time $n+1$.

(ii) Suppose that $R_n = r$ (a constant), $n = 0, 1, \dots, N-1$. At each time $n < m$, the long position consists of $(1+r)^{m-n-1}$ such forward contracts maturing at m. Show that the total payoff from these contracts is

$$\mathrm{Fut}_{n+1,m} - \mathrm{Fut}_{n,m}.$$

Confirm this using the risk-neutral pricing formula.

Problem 3. *(Markov property from quadratic variation in symmetric random walk)*

Let $M_0, M_1, M_2, \ldots$ denote a symmetric random walk with $M_0 = 0$, $M_{n+1} = M_n + X_{n+1}$, $n = 0, 1, 2, \ldots$, where $X_n \in \{-1, +1\}$ are i.i.d. binary random variables with $\mathbb{P}[X_n = +1] = \mathbb{P}[X_n = -1] = 1/2$. Consider the process

$$I_n = \sum_{j=0}^{n-1} M_j\left(M_{j+1} - M_j\right), \qquad n = 1, 2, \ldots, \qquad I_0 = 0.$$

(i) Show that

$$I_n = \frac{1}{2}\left(M_n^2 - n\right).$$

(ii) Let $f(\cdot)$ be any real-valued function. Let $g(\cdot)$ be defined by the conditional expectation

$$g(I_n) = \mathbb{E}[f(I_{n+1})|\mathcal{F}_n] = \mathbb{E}_n[f(I_{n+1})], \qquad n = 1, 2, \ldots .$$

- (a) For every such function $f(\cdot)$, show that the function $g(\cdot)$ exists and derive an explicit expression for it.
- (b) Conclude that $(I_n)_{n\geq 0}$ is a Markov process.

Problem 4. *(Itô's formula and SDE for transformed geometric Brownian motion)*

Let S_t be a geometric Brownian motion given by the stochastic differential equation (SDE)

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and W_t is a standard Brownian motion. Let $Y_t = g(S_t, t)$ for some twice continuously differentiable function $g: \mathbb{R}^+ \times [0, T] \to \mathbb{R}$.

(i) State Itô's formula for the function $g(S_t, t)$.

(ii) Let $g(x) = x^p$, with $p > 0$. Let $Y_t = S_t^p$. Use Itô's formula to derive the SDE satisfied by Y_t.