**Numerical Analysis Preliminary Examination**
**August 2024**

# INSTRUCTIONS

Write your assigned number, the problem number and the page number on every page that you submit.

Submit **exactly 8 problems**, from the problems below, to be scored.

# PROBLEMS

**Problem 1** Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite, i.e. $A^T = A$ and $\mathbf{x}^T A\mathbf{x} > 0$ for all non-zero vectors $\mathbf{x} \in \mathbb{R}^n$. Consider the problem of solving $A\hat{\mathbf{x}} = \mathbf{b}$ where $\mathbf{b} \in \mathbb{R}^n$ is given. Let $D_A \in \mathbb{R}^{n\times n}$ be the $n \times n$ diagonal matrix whose diagonal entries are the same as the diagonal entries of $A$ and whose off-diagonal entries are zero.

1. Prove that the diagonal entries of $D_A$ are all positive.
2. Prove that all of the eigenvalues of $A$ are positive.
3. Prove that there exists an $\omega > 0$ such that the Jacobi over relaxation (JOR) method

$$\mathbf{x}_{k+1} = (I - \omega D_A^{-1} A)\mathbf{x}_k + \omega D_A^{-1}\mathbf{b}$$

converges to the solution $\hat{\mathbf{x}}$. You may use the fact that if $C$ and $B$ are symmetric, positive definite $n \times n$ matrices, then $CB$ has positive eigenvalues.

**Problem 2** An $n \times n$ matrix $P$ is an orthogonal projector if $P^2 = P$ and $P = P^*$ where $P^*$ is the conjugate transpose of $P$. Prove that if $P$ is an orthogonal projector, the singular values $\sigma_1 \geq \sigma_2 \geq .. \geq \sigma_n \geq 0$ of $P$ must satisfy, for each individual value of $1 \leq j \leq n$, $\sigma_j = 1$ or $\sigma_j = 0$. You may use the fact that the singular values of a matrix are unique.

**Problem 3** Consider the two matrices below

$$A_1 = \begin{bmatrix} 2 & 0 \\ 1\times 10^6 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 2 & 1\times 10^{-5} \\ 1\times 10^6 & -1 \end{bmatrix}$$

Note that $A_2 = A_1 + \delta A_1$ where

$$\delta A_1 = \begin{bmatrix} 0 & 1\times 10^{-5} \\ 0 & 0 \end{bmatrix}.$$

Let $\mathcal{X} \subset \mathbb{R}^2$ be the set of $2 \times 2$ matrices that have a dominant eigenvalue, i.e. $|\lambda_1| > |\lambda_2|$. Consider the problem $F : \mathcal{X} \to \mathbb{R}$ mapping $A \in \mathcal{X}$ to its dominant eigenvalue, $F(A) = \lambda_1$. Note that $F(A_1) = 2$ is clear.

1. Start with the vector $\mathbf{v}_0 = [5.25 \times 10^{-6}, 1]^T$. Use one step of the power method to approximate the dominant eigenvalue of $A_2$, above.

2. Use the matrix $\infty$-norm, $\|A\|_\infty = max\,\{|A_{11}| + |A_{12}|, |A_{21}| + |A_{22}|\}$ on $\mathbb{R}^{2\times 2}$, the absolute value on $\mathbb{R}$ and the results of step (1), above to report the value that you would get if you used $\delta A_1$ to estimate the *absolute condition number* of the problem $F$ at $A_1$.

**Problem 4** Compute a full QR decomposition for the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**Problem 5** Consider the initial value problem

$$\begin{aligned} y'(t) &= y(t) \\ y(0) &= 1 \end{aligned}$$

Let $h = t_{n+1} - t_n$ be a uniform time step size for the solving the initial boundary value problem, above, on some interval $[0, T]$ with the implicit Runge-Kutta method defined below

$$\begin{array}{c|cc} 1 & 1 & 2 \\ 2 & 1 & 1 \\ \hline & \frac{3}{4} & \frac{1}{4} \end{array}$$

1. Show that the 2-stage implicit Runge-Kutta method applied to the initial value problem, both specified above, can be written in the form

$$L_h \begin{bmatrix} K_1 \\ K_2 \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} u_n \\ u_n \\ u_n \end{bmatrix},$$

where $L_h$ is a $3 \times 3$ matrix whose entries depend on the time step-size $h$

2. Let $h = 1/4$ and compute the LU decomposition of the matrix $L_h$ from part (1), above. Show each step of your process.

**Problem 6** The trace of an $n \times n$ matrix is defined to be the sum of its diagonal entries, i.e. $\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}$.

1. Let $\mathcal{V} = \mathbb{R}^{m\times n}$ be the real vector space of $m \times n$ matrices with real valued entries. Addition of matrices and multiplication by a (real) scalar are defined in the usual way. Show that $\dim(\mathcal{V}) = mn$
2. Show that $\langle A, B\rangle = \mathrm{tr}(B^T A)$ defines a (real) inner product on $\mathbb{R}^{n\times n}$.
3. Let $m = n = 2$ and define

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}.$$

   Find a matrix $B \in \mathbb{R}^{2\times 2}$ that satisfies the following conditions:

   - $B$ is $\langle\cdot,\cdot\rangle$-orthogonal to $A$ (i.e. with respect to the inner product defined above)
   - There exists a real number $\eta \in \mathbb{R}$ such that $I_2 = B + \eta A$ where $I_2$ is the $2 \times 2$ identity matrix

**Problem 7** Consider the set of data points $\mathbf{x}_i = (x_i, y_i) \in \mathbb{R}^2$, for $1 \le i \le 6$, specified by

$$(0, 10) \quad (1, 8) \quad (-3, 3) \quad (-4, 0) \quad (2, 3) \quad (-1, 2)$$

Find the line $l(x) = ax + b$ minimizing $\left(\sum_{i=1}^{6} (l(x_i) - y_i)^2\right)^{1/2}$.

**Problem 8** Consider the interval $[0, 1]$ and the elliptic boundary value problem (BVP) defined by

$$-\partial_{xx} u = x, \qquad u(0) = u(1) = 0$$

1. Define the (discrete) test space (which is the same as the discrete trial space for this problem) needed to discretize the BVP, above, with linear Lagrange finite elements using a uniform partition of the interval $[0, 1]$ with size $h = 1/N$.
2. Write the corresponding variational formulation for solving the BVP, above, with the test and trial spaces from part (1).
3. Solve the variational problem, from part (2) with $N = 3$, for the discrete solution $u_h(x)$.

**Problem 9** Let $\xi \in (\frac{-\pi}{2}, \frac{\pi}{2})$ and define a quadrature rule on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ by

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\,dx \approx \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[L_2^\xi(f)\right](x)\,dx$$

where $\left[L_2^\xi(f)\right](x)$ is the *quadratic* Lagrange interpolant to $f(x)$ based on the (unique) nodes $x_0 = \frac{-\pi}{2}$, $x_1 = \xi$ and $x_2 = \frac{\pi}{2}$, with $x_0 < \xi < x_1$. For a fixed function, $f$, the quadrature error is a function of $\xi$ defined by

$$E_f(\xi) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\,dx - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[L_2^\xi(f)\right](x)\,dx$$

1. Determine $E_f(\xi)$ for $f(x) = cos(x)$. Note that your answer will be a function in terms of $\xi$.
2. Based on your result, above, show that finding a value for the node $\xi$ so that the quadrature rule integrates $f(x) = cos(x)$ exactly can be done by solving an equation of the form $\phi(\xi) = 0$ where $\phi(\xi) = p(\xi) + \alpha cos(\xi)$ with $p(\xi)$ a (quadratic) polynomial and $\alpha \in \mathbb{R}$.
3. Using the initial iterate $\xi_0 = \frac{\pi}{4}$, take one step of Newton's method to estimate the value $\widehat{\xi}$ yielding $\phi(\widehat{\xi}) = 0$.

**Problem 10** Let $(M, \mathcal{F}, \epsilon)$ denote a computing machine $M$ that uses a floating point system $\mathcal{F}$ with finite precision; the value $\epsilon > 0$ denotes machine epsilon for the device. We assume that the machine $(M, \mathcal{F}, \epsilon)$ has the following two properties:

- $M$ can represent any $x \in \mathbb{R}$ by a floating point representation, $\text{fl}(x) \in \mathcal{F}$ that satisfies $\text{fl}(x) = x(1 + \hat{\epsilon})$ for some $|\hat{\epsilon}| \leq \epsilon$.
- $M$ implements the addition of real values in a hardware process that is represented by the symbol $\oplus$ and satisfies $x \oplus y = \text{fl}(x + y)$ for $x$, $y \in \mathbb{R}$.

Consider the function $g(x) = 2x$. An *algorithm* for computing $g(x)$, on the computer $M$, can be defined by $\tilde{g}(x) = \text{fl}(x) \oplus \text{fl}(x)$. Prove that $\tilde{g}(x)$ is a backward stable algorithm for $g(x)$.