Instructions

Do all problems. For clarity and to assist the committee in grading, please write as neatly as possible to avoid any misunderstandings. You are allowed to use a single non-graphing, non-programmable calculator.

Condition numbers (20 pts)

Problem 1 Let $\mathbb{A}, \mathbb{B}$ be two non-singular square matrices in $\mathbb{R}^{n\times n}$. Under the infinity norm, prove or disprove the following claim: "It always holds true that $\operatorname{cond}(\mathbb{A}\mathbb{B}) = \operatorname{cond}(\mathbb{A})\operatorname{cond}(\mathbb{B})$". **(20 pts)**

Linear spaces (70 pts)

Problem 2 Let $\mathbb{P}_m$ be the (linear) space containing all polynomials of degree $\leq m$. Show that the class of rational functions,

$$\mathbb{R}_{r,s} = \{\varphi : \varphi = p/q, p \in \mathbb{P}_r, q \in \mathbb{P}_s\},$$

is not a linear space. **(20 pts)**

Problem 3 Consider the L_2 norm and inner product with discrete measure. Recall that the discrete measure associated with the point set $\{t_1, t_2, \dots, t_N\}$ is a measure $d\lambda$ that is nonzero only at t_i and has the value $w_i > 0$ there.

a) **(20 pts)** Prove that for any $u(t), v(t)$ with $\|u\|_2 = \|v\|_2 = 1$ and for all real numbers a, b, we have

$$\|au - bv\|_2 = \|bu - av\|_2.$$

b) **(30 pts)** Apply this identity to prove that the distance function:

$$d(x,y) = \frac{\|x-y\|_2}{\|x\|_2\|y\|_2}, \quad \|x\|_2, \|y\|_2 > 0.$$

is a metric. You may find the choices $u = \frac{x}{\|x\|_2}, v = \frac{y}{\|y\|_2}, a = \|y\|_2, b = \|x\|_2$ useful.

Interpolation (50 pts)

Problem 4 Let $\{x_0, x_1, \dots, x_n\}$ be the $n+1$ distinct points on $[a,b]$. Construct a quadratic function of the form,

$$QS_i(x) = a_i(x-x_i)^2 + b_i(x-x_i) + c_i, \quad x \in [x_i, x_{i+1}],$$

using the usual interpolation and continuity conditions. In addition, we set $b_0 = 0$ as an additional condition. Explain why the extra condition is needed. **(50 pts)**

Numerical differentiation (110 pts)

Problem 5
Let $x_i = x_0 + ih$ $(i = 0, 1, 2)$.

a) Apply the differentiation by interpolation method to derive the three-point endpoint formula:

$$f'(x_2) = \frac{1}{2h}\left[f(x_0) - 4f(x_1) + 3f(x_2)\right] + \frac{h^2}{3} f'''(\xi(x_2)),$$

where $\xi = \xi(x_2) \in (x_0, x_2)$. **(30 pts)**

b) Let $h = 0.1$, and consider $f(x) = e^x$. Apply the above three-point endpoint formula to find $\xi(x_2)$ explicitly. **(20 pts)**

c) Assume that $M = \max\left\{|f(x)|, \left|f'''(x)\right|\right\}$. Suppose that the actual values $f_\varepsilon(x_i)$ are given in the multiplicative form:

$$f_\varepsilon(x_i) = f(x_i)(1 + \varepsilon_i).$$

Let $\varepsilon = \max\{|\varepsilon_i|\}$. Then, derive an upper bound of

$$\left| f'(x_2) - \frac{1}{2h}\left[f_\varepsilon(x_0) - 4f_\varepsilon(x_1) + 3f_\varepsilon(x_2)\right] \right|$$

in terms of h and ε. **(30 pts)**

Problem 6 For $x \in [0, 1]$, consider computing

$$u(x) = f''(x), \quad f \in C^2.$$

Consider the L_2 norm of the form $\|v\|_2^2 = \int_0^1 |v(x)|^2 \, dx$. Assume that $f(x)$ is perturbed to $f_\varepsilon(x)$ by:

$$f_\varepsilon(x) = f(x) + \varepsilon \sin\left(\frac{\pi x}{\varepsilon}\right), \quad \varepsilon \in (0, 1).$$

This leads to the corresponding problem of solving $u_\varepsilon(x) = f''_\varepsilon(x)$.

a) Prove that $\lim_{\varepsilon \to 0^+} \|f_\varepsilon - f\|_2^2 = 0$. **(10 pts)**

b) Prove that $\lim_{\varepsilon \to 0^+} \|u_\varepsilon - u\|_2^2$ is infinite. You may find the identity $2\sin^2(\theta) = 1 - \cos(2\theta)$ useful. **(20 pts)**

Numerical integration (70 pts)

Problem 7 For fixed numbers $a, b \in \mathbb{R}$, let $I(z) = g(z) + \int_a^z f(x)\,dx$ for some $z \in (a,b)$. Assuming that $f \in C^2[a,b], g \in C^1[a,b]$, for a fixed $z = z_0$, we apply the trapezoidal rule $T_h(f)(z_0) = \frac{h}{2}[f(a) + f(z_0)]$ to approximate the integral in $I(z)$.
Let $z_0^\varepsilon \in (a,b)$ be a perturbation of z_0 with $z_0^\varepsilon = z_0 + \varepsilon$ for some $|\varepsilon| \in (0,1)$. Accordingly, we get h^ε instead of h in the approximation.

a) **(20 pts)** Let $K(f)(z_0) = \int_a^{z_0} f(x)\,dx$.
Let $M = \max_{x\in[a,b]} \left\{\left|f''(x)\right|, \left|f'(x)\right|, |f(x)|, |g(x)|\right\}$. Show that

$$|K(f)(z_0) - T_h(f)(z_0)| \leq \frac{M}{12}h^3.$$

b) **(20 pts)** Let $T_{h^\varepsilon}(f)(z_0^\varepsilon) = \frac{h^\varepsilon}{2}[f(a) + f(z_0)]$. Prove that

$$\left|T_{h^\varepsilon}(f)(z_0^\varepsilon) - T_h(f)(z_0)\right| \leq |\varepsilon|\,M + \frac{h}{2}M|\varepsilon|.$$

c) **(10 pts)** Using the above estimations, find an upper bound of

$$\left|I(z_0) - (g(z_0^\varepsilon) + T_{h^\varepsilon}(f)(z_0^\varepsilon))\right|$$

in terms of $|\varepsilon|$ and h.

Problem 8 **(20 pts)** Assume that the Turán quadrature formula:

$$\int_a^b f(x)\,w(x)\,dx = \sum_{i=0}^{n}\left[w_i f(x_i) + w_i' f'(x_i) + w_i'' f''(x_i)\right] + E_n(f)$$

has degree of exactness $d = 4n+3$. Then, prove that the following conditions are satisfied:

a) $E_n(f) = 0$ if $f \in \mathbb{P}_{3n+2}$.

b) The node polynomial $\pi_n(x) = \prod_{i=0}^{n}(x - x_i) \in \mathbb{P}_{n+1}$ satisfies

$$\int_a^b [\pi_n(x)]^3\,p(x)\,w(x)\,dx = 0 \quad \text{for all } p \in \mathbb{P}_n.$$

You may find the fact that $[\pi_n]^3 p \in \mathbb{P}_{4n+3}$ for any $p \in \mathbb{P}_n$ useful.

Approximations of nonlinear equations (80 pts)

Problem 9

a) Prove that $f(x) = e^x (1-x)$ maps from $[0,1]$ to $[0,1]$. **(20 pts)**

b) Prove that $f(x)$ does not satisfy the condition $\left|f^{'}(x)\right| < 1$ in the existence and uniqueness theorem for a fixed point. However, prove that there still exists a unique fixed point of $f(x)$ on $[0,1]$. **(20 pts)**

Problem 10 Suppose that $f \in C^2[a,b]$ with $L = \max_{x\in[a,b]} \left|f^{''}(x)\right|$, $M_1 = \min_{x\in[a,b]} \left|f^{'}(x)\right| > 0, M_2 = \max_{x\in[a,b]} \left|f^{'}(x)\right|$ and $N = \max_{x\in[a,b]} |f(x)| > 0$. Consider the Newton's method for approximating $f(x) = 0$. Let $x^* \in (a,b)$ be a unique root such that $f^{'}(x^*) \neq 0$. Assume that the actual value of the initial guess $p_0 \in (a,b)$ (with $|p_0 - x^*| < \delta$ for some sufficiently small δ) is $p_0^\alpha \in (a,b)$ satisfying $f^{'}(p_0^\alpha) \neq 0$ and

$$\left|p_0^\alpha - p_0\right| \leq \alpha, \quad \alpha \in (0,1).$$

Let $p_1 = p_0 - \frac{f(p_0)}{f^{'}(p_0)}$ and $p_1^\alpha = p_0^\alpha - \frac{f(p_0^\alpha)}{f^{'}(p_0^\alpha)}$.

a) **(10pts)** By considering $g(t) = f(y + t(x-y))$ for $t \in [0,1]$ and $x, y \in [a,b]$, prove that for any $x \in [a,b]$ close to p_0,

$$\left|f(x) - f(p_0) - f^{'}(p_0)(x - p_0)\right| \leq \int_0^1 \left|f^{'}(p_0 + t(x-p_0)) - f^{'}(p_0)\right| |x - p_0| \, dt.$$

b) **(10pts)** Using the above estimate with $x = x^*$, prove that

$$|p_1 - x^*| \leq \frac{L}{2M_1}\delta^2.$$

c) **(20 pts)** Estimate $\left|p_1^\alpha - x^*\right|$ in terms of $\delta, \alpha, L, M_1, M_2, N$.