May 2026 Numerical Analysis Prelim Exam

1. [**10 points**] Suppose that $N(h)$ is an approximation to M for every $h > 0$ and that

$$M = N(h) + K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots$$

for some constants $K_1, K_2, K_3, \cdots$.

(a) [5 points] Use the values $N(h), N(h/2)$ and $N(h/4)$ to produce an $O\left(h^6\right)$ approximation to M.

(b) [5 points] Given

$$N(h) = 1.570796, \quad N(h/2) = 1.896119, \quad N(h/4) = 1.974232.$$

Construct the best approximation to M as you can.

2. [**10 points**] Consider the definite integral

$$I = \int_a^b f(x) dx$$

(a) [3 points] Construct an approximation to I by using the composite trapezoidal rule with a uniform partition of the interval $[a, b]$.

(b) [4 points] Suppose $f \in C^2[a, b]$, show that the above approximation is second-order accurate.

(c) [3 points] Suppose f is periodic and smooth on the interval $[a, b]$, show that the above approximation is spectral order accurate.

3. [**15 points**] Define an iteration formula by

$$x_{n+1} = z_{n+1} - \frac{f\left(z_{n+1}\right)}{f'\left(x_n\right)}, \quad z_{n+1} = x_n - \frac{f\left(x_n\right)}{f'\left(x_n\right)}, \quad n = 0, 1, 2, \ldots.$$

(a) [5 points] Assume $f(x), f'(x), f''(x)$, and $f'''(x)$ are continuous for all x in some neighborhood of α, and assume $f(\alpha) = 0, f'(\alpha) \neq 0$. Then if x_0 is chosen sufficiently close to α, show that the order of convergence of $\{x_n\}$ to α is at least 3.

(b) [5 points] For a given $a > 0$, based on the above iteration formula, design an iteration formula for computing $\sqrt{a}$.

(c) [5 points] Use your iteration formula to compute approximations of $\sqrt{10}$ with $x_0 = 3$ for three steps.

4. [**15 points**] Consider the 2×2 matrix

$$A = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.$$

(a) [5 points] Under what condition for ρ, the Gauss-Seidel iterative method is convergent?

(b) [5 points] Find the region of the relaxation parameter ω such that the SOR method converge. What is the optimal relaxation parameter?

(c) [5 points] Repeat the above two questions for the matrix

$$A = \begin{bmatrix} 1 & \rho \\ -\rho & 1 \end{bmatrix}.$$

5. [**20 points**] Consider the following two-point boundary value problem (BVP)

$$\begin{aligned} -u''(x) + \gamma(x)u(x) &= f(x), \quad 0 < x < 1, \\ u(0) = \alpha, \quad u(1) &= \beta; \end{aligned}$$

where $\gamma(x)$ is a given non-negative function. Choose mesh size $h = \frac{1}{M}$ and denote $x_j = jh$ for $j = 0, 1, \ldots, M$. In addition, let u_j be the numerical approximation of $u(x_j)$ for $j = 0, 1, \ldots, M$.

(a) [10 points] Construct a second order finite difference discretization for the above BVP and prove the following error estimate

$$\left\| u - u^h \right\|_\infty := \max_{0 \le j \le M} |u(x_j) - u_j| \le Ch^2,$$

where C is a constant independent of h.

(b) [10 points] Construct a fourth order compact finite difference discretization for the above BVP and establish the following error estimate

$$\left\| u - u^h \right\|_\infty := \max_{0 \le j \le M} |u(x_j) - u_j| \le Ch^4,$$

where C is a constant independent of h.

6. [**20 points**] Consider the following gradient flow

$$\begin{aligned}
&\partial_t u(x,t) = \partial_{xx} u(x,t) - V(x)u - f\left(|u|^2\right) u, \quad 0 < x < 1, \quad t > 0, \\
&u(x,0) = g_0(x), \quad 0 \le x \le 1, \\
&u(0,t) = u(1,t) = 0, \quad t \ge 0,
\end{aligned}$$

where $u = u(x,t)$ is a real-valued function, $V(x) \ge 0$ for $0 \le x \le 1$ is a given function and $f(\rho) \ge 0$ is a function of ρ.

(a) [5 points] Define the mass and energy as

$$\begin{aligned}
M(t) &:= \int_0^1 |u(x,t)|^2 dx, \quad t \ge 0 \\
E(t) &:= \int_0^1 \left[|\partial_x u|^2 + V(x)|u|^2 + F\left(|u|^2\right) \right] dx
\end{aligned}$$

where

$$F(\rho) = \int_0^\rho f(s) ds$$

Show that the mass and energy are diminishing, namely,

$$M(t_2) \le M(t_1), \quad E(t_2) \le E(t_1), \quad 0 \le t_1 < t_2.$$

(b) [5 points] Construct a second-order (in space and time) finite difference method for the problem such that the mass and energy are diminishing in the discretized level.

(c) [5 points] Find the local truncation error of your method. Is the method consistent?

(d) [5 points] When $f(\rho) = 0$, prove the error estimate by using the energy method.

7. [**10 points**] Proof the following statements.

(a) [5 points] Let $h = 1/n$ and $A = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ 0 & 0 & & 1 & -2 & 1 \\ 1 & 0 & & 0 & 1 & -2 \end{pmatrix} \in \mathbb{R}^{n \times n}$.

Prove that the spectral radius of A is bounded above by $4/h^2$.

(b) [5 points] Suppose $A = (a_{i,j}) \in \mathbb{C}^{n \times n}$ is an invertible matrix with $\sum_{j=1}^n |a_{i,j}| = b$, for any $i \in \{1, 2, \ldots, n\}$. Show, first that if D is an invertible diagonal matrix, then $\|DA\|_\infty = b\|D\|_\infty$ and use this to show that

$$\mathrm{cond}_\infty(A) \le \mathrm{cond}_\infty(DA),$$

where $\mathrm{cond}_\infty(A) = \|A\|_\infty \left\|A^{-1}\right\|_\infty$.