Numerical Analysis

Preliminary Examination

May 1996

Do seven of the following ten problems. Clearly indicate which seven problems are to be graded.

1. In producing cubic interpolating splines it is necessary to solve the linear system $Ax = b$ with

$$A = \begin{bmatrix} \frac{h_1}{3} & \frac{h_1}{6} & 0 & \cdots & 0 \\ \frac{h_1}{6} & \frac{h_1+h_2}{3} & \frac{h_2}{6} & & \vdots \\ \vdots & & \ddots & & \\ & & \frac{h_{m-1}}{6} & \frac{h_{m-1}+h_m}{3} & \frac{h_m}{6} \\ 0 & \cdots & 0 & \frac{h_m}{6} & \frac{h_m}{3} \end{bmatrix}$$

where $h_i > 0, i = 1 \ldots m$. Show that A is nonsingular and that

$$\frac{1}{6} \min_{1 \le i \le m} (h_i) \le |\lambda| \le \max_{1 \le i \le m} (h_i)$$

for all eigenvalues of A.

2. Assume that you are using the numerical approximation

$$f''(x_1) \approx D_h^{(2)} f(x_1) = \frac{f(x_2) - 2f(x_1) + f(x_0)}{h^2}$$

with $x_j = x_0 + jh$ for $f \in C^2$. Also assume that you have empirical data $\tilde{f}_i$ for which

$$f(x_i) = \tilde{f}_i + \epsilon_i \qquad i = 0, 1, 2$$

so that the actual numerical derivative computed is

$$\tilde{D}_h^{(2)} f(x_1) = \frac{\tilde{f}_2 - 2\tilde{f}_1 + \tilde{f}_0}{h^2}.$$

Assume that the errors are such that $-E \le \epsilon_i \le E$ and find $c_1$ and $c_2$ for which

$$\left| f''(x_1) - \tilde{D}_h^{(2)} f(x_1) \right| \le c_1 h^2 + \frac{c_2 E}{h^2}.$$

3. Suppose $y''(t) = t \ln t + \frac{2}{t} y'(t) - \frac{2}{t^2} y(t)$ for $1 \le t \le 2$ with $y(1) = 1$ and $y'(1) = 0$. Convert this second-order problem to a system of first-order equations and compute $\vec{y}_k$ for $k = 1$ and 2 using Euler's method with step length $h = 0.1$.

9. Show that if $f$ and $g$ are polynomials of degree less than $n$, if $x_i$, $i = 1, 2, \cdots, n$, are the roots of the $n$th Legendre polynomials, and if

$$\gamma_i = \int_{-1}^{1} L_i(x)dx$$

with

$$L_i(x) = \prod_{k \neq i, k=1}^{n} \frac{x - x_k}{x_i - x_k}, \quad i = 1, 2, \cdots, n,$$

then

$$\int_{-1}^{1} f(x)g(x)dx = \sum_{i=1}^{n} \gamma_i f(x_i)g(x_i).$$

Notice that $\int_{-1}^{1} L_i(x)dx = \int_{-1}^{1}(L_i(x))^2 dx$.

10. (a) Use the trigonometric identity $\sin(\alpha - \beta) + \sin(\alpha + \beta) = 2\sin\alpha\cos\beta$ to show that:

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \sin(k\pi/4) \\ \sin(2k\pi/4) \\ \sin(3k\pi/4) \end{pmatrix} = 2(1 - \cos\frac{k\pi}{4}) \begin{pmatrix} \sin(k\pi/4) \\ \sin(2k\pi/4) \\ \sin(3k\pi/4) \end{pmatrix}, \quad k = 1, 2, 3.$$

(b) Find the condition number in the $L_2$-norm (i.e., $\|A\|_2\|A^{-1}\|_2$) of the above tridiagonal matrix that occurs in the numerical solution of differential equations.

(c) Generalize this result to the $n \times n$ tridiagonal matrix for which each diagonal element is 2, and each super-diagonal and sub-diagonal element is -1.

4. Consider a descent method for approximating the minimum of a function. Let

(a) $f : R^m \to R^1$ be a $C^1$ function.

(b) $\phi_n(t) = f(x_n - t\nabla f(x_n))$.

(c) $t_n = \min_{t\in\Re^+} (\phi_n(t))$

(d) $x_{n+1} = x_n - t_n \nabla f(x_n)$.

Show that either $\nabla f(x_n) = 0$ or $f(x_{n+1}) < f(x_n)$. (Consider the behavior of $\phi_n(t)$ near $t = 0$).

5. Let $(x_i, y_i)$, $i = 1, 2, \ldots, N$ be N points with $N \geq M$ and $a \leq x_1 < x_2 < x_3 < \cdots < x_N \leq b$. Let $S = span\,(\phi_1(x), \phi_2(x), \ldots, \phi_M(x))$ where $\{\phi_i(x)\}_{i=1}^M$ are linearly independent functions in $C\,[a,b]$, and let $g(x) = \sum_{i=1}^M c_i\phi_i(x) \in S$ satisfy $\sum_{j=1}^N (g(x_j) - y_j)^2 \leq \sum_{j=1}^N (f(x_j) - y_j)^2$ for any $f \in S$.
Find an $M \times M$ linear system satisfied by the constants $c_1, c_2, \ldots, c_M$.

6. Suppose that $f \in C^1[0,1]$ satisfies $f(x) = \int_0^1 f(y)g(y,x)dy + b(x)$ where $g \in C^1([0,1] \times [0,1])$, $\max_{0\leq x,y\leq 1} |g(y,x)| = \lambda < 1$, and $b \in C[0,1]$. Consider the numerical scheme to approximate $f(x)$ at discrete points: $f_k = \sum_{m=0}^{N-1} f_m g(y_m, x_k)\Delta y + b(x_k)$ for $k = 0, 1, \cdots, N$, where $\Delta y = \frac{1}{N}$, $x_k = k\Delta y$ and $y_m = m\Delta y$. The value of $f_k$ approximates $f(x_k)$.

Show that $\max_{0\leq k\leq N} |f(x_k) - f_k| \leq \frac{1}{1-\lambda}\frac{\Delta y}{2} \max_{0\leq x,y\leq 1} \left|\frac{\partial}{\partial y}(f(y)g(y,x))\right|$.

(Use the fact that $\left|\int_0^1 r(y)dy - \sum_{m=0}^{N-1} r(y_m)\Delta y\right| \leq \frac{\Delta y}{2} \max_{0\leq y\leq 1} |r'(y)|$ for $r \in C^1[0,1]$).

7. Consider Newton's method $x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$. Suppose that:

(a) $f'(x) > 0$ for $x \in \Re$,

(b) $f(y) - f(x) \geq f'(x)(y - x)$, for $x, y \in \Re$,

(c) $f(0) < 0$, $x^{(0)} > 0$, and

(d) $f(x^{(0)}) > 0$.

Show that $x^{(k+1)} \to x^*$ as $k \to \infty$, $x^* \leq \cdots \leq x^{(k+1)} \leq x^{(k)} \leq \cdots \leq x^{(1)} \leq x^{(0)}$ and $x^*$ is the unique zero of $f(x)$ in $\Re$.

8. Given $n+1$ distinct points $x_0, x_1, \cdots, x_n$, let

$$\Psi_n(x) = \prod_{i=0}^{n}(x - x_i), \quad l_j(x) = \prod_{i\neq j, i=0}^{n} \frac{x - x_i}{x_j - x_i},$$

and let $w_j = \Psi_n'(x_j)^{-1}$. Show that the polynomial $p_n(x) = \sum_{j=0}^{n} f(x_j)l_j(x)$ interpolating $f(x)$ can be written as

$$\sum_{j=0}^{n} \frac{w_j f(x_j)}{x - x_j} \Big/ \sum_{j=0}^{n} \frac{w_j}{x - x_j}$$

provided $x$ is not a node point.