Numerical Analysis Preliminary Examination

May 1997

Do seven of the following nine problems. Clearly indicate which seven problems are to be graded.

1. Let $\{p_0, p_1, \cdots, p_n\}$ be a set of orthonormal polynomials on $[-1, 1]$ with respect to a weight function $w(x)$ and let q_n be the least-squares approximation of f from polynomials of degree n or less with respect to the weight function $w(x)$. Denote

$$\|f - q_n\|_w^2 = \int_{-1}^{1} [f(x) - q_n(x)]^2 w(x) dx.$$

Show that

$$\|f - q_n\|_w^2 = \|f\|_w^2 - \sum_{j=0}^{n} \lambda_j^2,$$

where

$$\lambda_j = \int_{-1}^{1} p_j(x) f(x) w(x) dx, \quad j = 0, 1, \dots n.$$

2. Consider the problem of approximation of $\int_a^b f(x)dx$. Suppose that $\{p_n(x)\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials with respect to the inner product $(g, h) = \int_a^b g(x)h(x)dx$ with each $p_n(x)$ of degree n. Let $\int_a^b f(x)dx \approx \sum_{j=0}^{N} \alpha_j f(x_j)$ where x_j, $j = 0, 1, \dots, N$, are the zeros of $p_{N+1}(x)$ and $\alpha_j = \int_a^b l_j(x)dx$ with $l_j(x) = \prod_{k\neq j, k=0}^{N} \frac{x - x_k}{x_j - x_k}$. Prove that $\int_a^b p(x)dx = \sum_{j=0}^{N} \alpha_j p(x_j)$ for any polynomial p of degree less than or equal to $2N + 1$. (Note that $p(x) = p_{N+1}(x)q(x) + r(x)$ where r and q are of degree N.)

3. Prove that for any matrix A the series $I + A + A^2 + \cdots$ converges if and only if the spectral radius of A is less than one.

4. Let $x_0, \dots, x_n$ be distinct real numbers and f be a continuous real function. Show that there is a unique function

$$p_n(x) = \sum_{j=0}^{n} c_j e^{jx}$$

that interpolates f at these numbers, i.e., $p_n(x_j) = f(x_j)$, for $j = 0, 1, 2, \dots, n$.

5. The Legendre polynomials may be defined in the following way: $L_0(x) = 1$, $L_1(x) = x$, and

$$(k+1)L_{k+1}(x) = (2k+1)xL_k(x) - kL_{k-1}(x), \quad k = 1, 2, \ldots \tag{1}$$

Show that

(a) $L_{2m}(x)$ are even functions and $L_{2m+1}(x)$ are odd functions for $m \geq 0$.

(b) $L_k(1) = 1$ for all $k \geq 1$.

(c) L_k has k distinct simple zeros in $(-1, 1)$ and the zeros of L_{k+1} are separated by those of L_k for $k \geq 1$.

6. Consider $\dfrac{dy}{dt} = f(t, y)$, $y(0) = \alpha$. Assume that f is continuous on $D = \{(t, y) : 0 \leq t \leq 1, -\infty < y < \infty\}$, the solution satisfies $|y''(t)| \leq M$ for $t \in [0, 1]$, and

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2| \quad \text{for } 0 \leq t \leq 1, \ y_1, y_2 \in R.$$

Show that the backward-Euler method $y_{i+1} = y_i + hf(t_{i+1}, y_{i+1})$ for $i = 0, 1, \ldots, N-1$ where $t_i = ih$, $y_0 = \alpha$, and $h = 1/N$ is convergent with order h, that is, $|y(t_i) - y_i| \leq Ch$ for $i = 0, 1, \ldots, N$.

(You may assume that $Lh < 1/2$ and note then that $(1 - hL)^{-1} \leq 1 + 2hL \leq e^{2hL}$. You may use the fact that

$$y(t_{i+1}) = y(t_i) + hf(t_{i+1}, y(t_{i+1})) - h^2 y''(\xi_i)/2$$

for some $\xi_i \in (t_i, t_{i+1})$.)

7. Let $x_k = kh$, $k = 0, 1, \ldots, N$, $h = 1/N$, define a partition of $[0, 1]$. Let

$$\phi_j(x) = \begin{cases} 1, & x_{j-1} \leq x \leq x_j \\ 0, & \text{otherwise} \end{cases}$$

for $j = 1, 2, \ldots, N$. Let PC be the set of piecewise constant functions expressed as $y(x) = \sum_{j=1}^{N} c_j \phi_j(x)$. Define $(f, g) = \int_0^1 f(x)g(x)dx$ and $\|g\|_2^2 = (g, g)$. Let $f \in C[0, 1]$. Find $y(x) = \sum_{j=1}^{N} c_j \phi_j(x)$ such that $\|f - y\|_2^2 \leq \|f - r\|_2^2$ for all $r \in$ PC. (You need to find the real numbers $c_1, c_2, \ldots, c_N$.)

8. (a) Show that if an $n \times n$ matrix A can be written as $A = LL^T$ where L is a nonsingular, real, lower triangular matrix, then A is real symmetric and positive definite.

(b) Let $A = \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}$. Find the Choleski decomposition of A.

9. Consider the sequence $\{x_n\}$ defined by $x_{n+1} = b + \epsilon g(x_n)$ for $n = 0, 1, 2, \ldots$ where $x_0 \in R$ is given. Assume that there is an $L > 0$ such that $|g(v) - g(w)| \leq L|v - w|$ for all $v, w \in R$. Find a condition on ϵ that will guarantee convergence of the sequence $\{x_n\}$ to the unique value x such that $x = b + \epsilon g(x)$.