Numerical Analysis Preliminary Examination 1998

Department of Mathematics and Statistics

Note: Do seven of the following nine problems. Clearly indicate which seven are to be graded.

1. Let $g \in C^2[a,b]$ and consider the fixed-point iteration $x_{j+1} = g(x_j)$ for $j = 0, 1, 2, \ldots$ with given $x_0 \in [a,b]$. Assume that $|g'(x)| \leq \alpha < 1$ for $x \in [a,b]$ and if $x \in [a,b]$ then $g(x) \in [a,b]$.

(a) Prove that the sequence $\{x_j\}_{j=0}^{\infty}$ is a Cauchy sequence and hence converges to some $\hat{x} \in [a,b]$ with $\hat{x} = g(\hat{x})$.

(b) Prove that if $g'(\hat{x}) = 0$, then $|x_{j+1} - \hat{x}| \leq \beta |x_j - \hat{x}|^2$, $j = 0, 1, 2, \ldots$ for some $\beta > 0$.

2. Consider the quadrature formula

$$(b-a)\sum_{j=0}^{m} f(x_j)\alpha_j \approx \int_a^b \rho(x) f(x)\,dx$$

where $\rho(x) > 0$ is a given weight function. Assume that the quadrature formula is exact for polynomials of degree $\leq 2m+1$. Prove that the weights α_j satisfy

$$\alpha_j = \frac{1}{b-a}\int_a^b \rho(x) L_j^2(x)\,dx, \quad j = 0, 1, \ldots, m$$

where

$$L_j(x) = \prod_{i=0, i\neq j}^{m} \frac{x - x_i}{x_j - x_i}.$$

3. Apply the Backward Euler method to $x' = f(t,x)$ with initial error $e_0 = x(0) - x_0$. Assume that $-m \leq f_x(t,x) \leq 0$ and $\|x''\|_\infty \leq M$. Derive an estimate for $e_n = x(t_n) - x_n$. (Note that $x_{n+1} = x_n + hf(t_{n+1}, x_{n+1})$ where $h = t_{n+1} - t_n$.)

4. Prove that the Jacobi iteration method for solving $A\vec{x} = \vec{b}$ converges for any 2×2 symmetric positive definite matrix. (Hint: Consider the eigenvalues of the Jacobi iteration matrix $J = -D^{-1}(A - D)$ where $D = \mathrm{diag}(a_{11}, a_{22})$.)

5. Assume that the function $f(x, y)$ has a unique minimum in the square $-1 \le x, y \le 1$ and $f \in C^1([-1,1] \times [-1,1])$. Describe the method of steepest descent and explain how you would implement the method to find the minimum of f on the square.

6. Assume that $f \in C^2[a,b]$. Let $M = \max_{a \le x \le b} |f''(x)|$.

(a) Prove that

$$\left| \int_a^b f(x)dx - (b-a) f\left(\frac{a+b}{2}\right) \right| \le (b-a)^3 MC$$

where C is a constant.

(b) Prove that

$$\left| \sum_{j=0}^{N-1} \left[\int_{a_j}^{a_{j+1}} f(x)dx - \frac{b-a}{N} f\left(\frac{a_j + a_{j+1}}{2}\right) \right] \right| \le \frac{(b-a)^3 MC}{N^2}$$

where $a_j = (b-a)j/N + a$ for $j = 0, 1, \ldots, N$ where C is the constant from part (a).

7. Let $f \in C[0, 2\pi]$ and $x_k = 2k\pi/N$ for $k = 0, 1, \ldots, N-1$. Let

$$p(x) = \sum_{j=0}^{N-1} c_j e^{ijx}, \quad \text{where } c_j = \frac{1}{N} \sum_{k=0}^{N-1} f(x_k) e^{-ijx_k}.$$

Prove that $p(x_l) = f(x_l)$ for $l = 0, 1, \ldots, N-1$ where $i = \sqrt{-1}$.

8. Show that

$$\left| e^x - \frac{1 + x/2}{1 - x/2} \right| \le C|x^3|$$

for some constant $C > 0$ and $-1 \le x \le 1$.

9. Suppose that the elements of an $n \times n$ matrix A satisfy $a_{ij} \le 0$ if $i \ne j$ and $a_{ij} > 0$ if $i = j$. Let $D = \text{diag}(a_{11}, a_{22}, \cdots, a_{nn})$ and $B = D^{-1}(D - A) = I - D^{-1}A$. (Note that $B \ge 0$ and $A = D(I - B)$.) Suppose that the spectral radius $\rho(B) < 1$. Show that A is nonsingular and $A^{-1} \ge 0$.