Numerical Analysis Preliminary Examination 1999

Department of Mathematics and Statistics

Note: Do seven of the following nine problems. Clearly indicate which seven are to be graded.

1. Let the $m \times n$ $(m \geq n)$ matrix A have full rank. Show that

$$\begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

has a solution x where x minimizes $\|Ax - b\|_2$.

2. Consider the initial-value problem $y'(t) = f(t, y)$, $0 \leq t \leq 1$, $y(0) = y_0$. Suppose that $\max_{0 \leq t \leq 1} |y'''(t)| = M < \infty$ and

$$|f(t,u) - f(t,v)| \leq L|u - v|, \quad \left|\frac{df}{dt}(t,u) - \frac{df}{dt}(t,v)\right| \leq L^2|u - v|, \quad 0 \leq t \leq 1$$

for constants L and M. Let $h = 1/N$ and

$$y_{j+1} = y_j + hf(t_j, y_j) + \frac{h^2}{2}\frac{df}{dt}(t_j, y_j), \quad j = 0, 1, \ldots, N.$$

Prove that

$$\max_{0 \leq j \leq N} |y_j - y(t_j)| \leq CMh^2$$

where C depends on L but is independent of h.

3. Consider the approximation

$$\int_{-1}^{1} f(x)dx \approx f(-a) + f(a), \quad 0 < a < 1.$$

(i) Prove that the error in this approximation is bounded by

$$\left(\frac{4}{3}a^3 - a^2 + \frac{1}{3}\right) \max_{-1 \leq x \leq 1} |f''(x)|.$$

(Hint: Consider the Lagrange interpolant to $f(x)$ through points $-a$ and a).

(ii) What is the optimal a in the sense that the numerical integration scheme is exact for polynomials of degree as high as possible?

4. Let $F(x) = x + f(x)g(x)$, where $f(r) = 0$ and $f'(r) \neq 0$. Find the precise conditions on the function g so that the iteration $x_{k+1} = F(x_k)$, $k = 0, 1, 2, \ldots$ will converge cubically to r if started near r. (Hint: Recall for cubic convergence, $F(r) = F'(r) = 0$.)

5. In the integral $\int_{-\infty}^{\infty} f(x)w(x)dx$, let the weight function $w(x) = e^{-x^2}$. The 2-point Gauss quadrature formula $w_0 f(x_0) + w_1 f(x_1)$ is exact for polynomials of degree up to three. Determine x_0, x_1, w_0, and w_1. Note that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}, \quad \int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

6. Consider the eigenvalue problem $Ax = \lambda x$. Assume that one simple eigenvalue, λ_0, of A is between 2.9 and 3.1 and the absolute values of all the other eigenvalues are outside the interval $(2,4)$. Explain how the inverse power method can be used to approximate λ_0 and explain how fast the algorithm converges.

7. Suppose that $u_h(x)$ is an order h^2 approximation to function $u(x)$ in the sense that

$$u_h(x) = u(x) + C_2(x)h^2 + C_4(x)h^4 + O(h^6)$$

where $C_2(x)$ and $C_4(x)$ are independent of the convergence parameter h.

(i) Clearly explain how to obtain an order h^4 approximation to u from u_h and $u_{h/2}$.

(ii) Clearly explain how to obtain an order h^6 approximation to u from u_{2h}, u_h, and $u_{h/2}$.

8. Let A be nonsingular, u and v be column vectors, and $A + uv^T$ be nonsingular. Show that

$$(A + uv^T)^{-1} = A^{-1} - (A^{-1}uv^T A^{-1})/(1 + v^T A^{-1} u).$$

Verify that $1 + v^T A^{-1} u \neq 0$.

9. Suppose that A and B are nonsingular, $\|A - B\|_2$ is "small", and we have a fast algorithm for solving $Ax = b$. We use the following iterative scheme for solving $By = c$.

(a) Use the fast algorithm to solve for y_0 where $Ay_0 = c$.

(b) Let $r_i = By_i - c$, $Ad_i = r_i$, and $y_{i+1} = y_i - d_i$, for $i = 0, 1, 2, \ldots$. At each iteration, d_i is obtained by the fast algorithm.

Prove that y_i converges to y if $\|I - A^{-1}B\|_2 < 1$.