Numerical Analysis Preliminary Examination 2000

Department of Mathematics and Statistics

Note: Do seven of the following nine problems. Clearly indicate which seven are to be graded.

1. Let $\phi_N(x)$ be the continuous piecewise linear interpolant of $f(x)$ at $N+1$ evenly spaced points on $[a,b]$. That is, $\phi_N(x_i) = f(x_i)$ for $i = 0, 1, \cdots, N$, where $x_i = ih + a, h = (b-a)/N$, and $\phi_N(x)$ is linear on each subinterval $[x_i, x_{i+1}]$. Suppose that $f \in C^1[a,b]$. Prove that $\|\phi_N - f\|_\infty \le ch\|f'\|_\infty$ for some constant $c > 0$.

2. Let P_5 be the set of polynomials of degree ≤ 5. Suppose that $p \in P_5$ satisfies:
$\int_0^1 (x^{50} - p(x))^2 dx \le \int_0^1 (x^{50} - q(x))^2 dx$ for all $q \in P_5$. Show how p can be found.

3. Show that if $\int_a^b f(x)dx = (b-a)\sum_{j=0}^{m} \alpha_j f(x_j)$ for any polynomial of degree less than or equal to m, then $\alpha_k = \frac{1}{(b-a)}\int_a^b \prod_{i=0, i\ne k}^{m} \frac{(x-x_i)}{(x_k-x_i)} dx$, for $k = 0, 1, \cdots, m$. (Assume that $a \le x_0 < x_1 < \cdots < x_m \le b$.)

4. Let $A = \begin{bmatrix} 50 & 2 & 1 \\ 2 & 20 & 1 \\ 1 & 1 & 10 \end{bmatrix}$. Find a good value for μ to ensure rapid convergence of the inverse power method $(A - \mu I)^{-1}\mathbf{x}_k = \mathbf{x}_{k+1}$ to the eigenvector corresponding to the smallest eigenvalue of A. Estimate the rate of convergence of the method.

5. Consider the iterative procedure $\mathbf{x}_{k+1} = B\mathbf{x}_k + \mathbf{c}$ where $A = F(I - B), A$ is an $n \times n$ nonsingular matrix and $\mathbf{c} = F^{-1}\mathbf{b}$. Suppose that $\|B\|_2 = \gamma < 1$.

   a. Prove that given $\varepsilon > 0$ there is an $N > 0$ such that $\|\mathbf{x}_{m+p} - \mathbf{x}_m\|_2 < \varepsilon$ for $p \ge 1$ when $m \ge N$.

   b. Prove that if $\mathbf{x}^*$ satisfies $A\mathbf{x}^* = \mathbf{b}$, then $\|\mathbf{x}_k - \mathbf{x}^*\|_2 \le \gamma^k \|\mathbf{x}_0 - \mathbf{x}^*\|_2$.

6. Consider approximating $f(x)$ where $f(x) = \int_0^1 f(t)g(x,t)dt + r(x)$.

   Suppose that $g \in C([0,1] \times [0,1]), r \in C([0,1])$, and $\max_{0 \le x,t \le 1} |g(x,t)| = \alpha < 1$. Let $f_{k+1}(x) = \int_0^1 f_k(t)g(x,t)dt + r(x)$, with $f_0(x) = 0$, define a sequence $\{f_k(x)\}_{k=1}^{\infty}$ of approximations to $f(x)$. Show that $\|f_{k+1} - f\|_\infty \le \alpha^k \|r - f\|_\infty$ for $k = 0, 1, \cdots$.

7. Use a geometric argument to derive the secant method for iterative solution of $f(x) = 0$ where $f : R^1 \to R^1$. Show that the secant method is equivalent to Newton's method when $f(x)$ is a linear function.

8. Suppose that we wish to solve the linear system $A\mathbf{x} = \mathbf{b}$, but instead we compute $\mathbf{y}$ that solves the system $A\mathbf{y} = \mathbf{b} + \mathbf{p}$ with $\|\mathbf{p}\|$ small. Obtain upper and lower estimates for $\|\mathbf{x} - \mathbf{y}\|/\|\mathbf{x}\|$ in terms of $\|\mathbf{p}\|/\|\mathbf{b}\|$ and the condition number $\|A\|\|A^{-1}\|$. (That is, relate the relative residual to the relative error.)

9. **a.** Explain how you would produce a quadrature formula for $\int_0^1 f(x)dx$ that uses 5 points and is exact for all polynomials of degree less than or equal to 9. (Explain how to find the quadrature points and weights but do not actually calculate them.)

   b. Use the error in Hermite interpolation to produce an error expression for this quadrature formula when $f \in C^{10}[0, 1]$.