Numerical Analysis Preliminary Examination 2001

Department of Mathematics and Statistics

Note: Do seven of the following nine problems. Clearly indicate which seven are to be graded.

1. Let $U\boldsymbol{x} = \boldsymbol{b}$ where U is an $n \times n$ nonsingular upper triangular matrix. The vector $\boldsymbol{x}$ can be computed using the algorithm $x_n = b_n/u_{n,n}$ and $x_k = (b_k - \sum_{j=k+1}^{n} u_{k,j}x_j)/u_{k,k}$ for $k = n-1, n-2, \cdots, 1$. Prove that this algorithm requires exactly $(n^2 - n)/2$ subtractions and additions. (Note that $\sum_{i=1}^{M} i = M(M+1)/2$.)

2. **(a)** Describe the equations that characterize the best quadratic approximation to $f(x) = x^{1/2}$ in $L^2[0,1]$. That is, find equations for α, β, γ that minimize

$$\int_0^1 (\alpha + \beta x + \gamma x^2 - f(x))^2 dx.$$

Find the best *linear* approximation to $f(x) = x^{1/2}$ in $L^2[0,1]$. (Find α and β.)

3. **(a)** Derive the Trapezoidal rule on interval $[a,b]$ and then the Composite Trapezoidal rule on this interval.

   (b) Derive the error in the Trapezoidal rule on interval $[a,b]$ for functions $f \in C^2[a,b]$.

   (c) Let $f(x) = x^{8/7}$ and let $T_n(f)$ denote the Composite Trapezoidal rule on [0,1] with spacing $h = 1/n$. Find

$$\lim_{n\to\infty} n^2 \left(\int_0^1 f(x)dx - T_n(f) \right).$$

4. **(a)** Describe the Inverse Power Method for a matrix that has distinct eigenvalues. Describe the standard error estimates for this method.

   (b) Let $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 4 & 8 & 1 & 1 \\ 0 & 0 & 10 & 3 \\ 0 & 0 & 3 & 10 \end{bmatrix}$.

   Describe the result of Inverse Power Iteration for the matrix $(A - 2I)^{-1}$. Determine the limiting eigenvalue and eigenvector and estimate the rate of convergence of the iteration method. (First, calculate the eigenvalues of A.)

5. Suppose that matrix A is nonsingular, $\boldsymbol{x}$ is the solution of $A\boldsymbol{x} = \boldsymbol{b}$, $\|A^{-1}\|_2 = 10^3$, and $\|A\|_2 = 10^2$. We wish to solve $B\boldsymbol{z} = \boldsymbol{b}$ where $B = A - C$ and $\|C\|_2 = 10^{-4}$.

(a) Prove that B is nonsingular.

(b) Find an upper bound on $\|\boldsymbol{x} - \boldsymbol{z}\|_2$ in terms of $\|\boldsymbol{x}\|_2$, that is, find $c > 0$ such that $\|\boldsymbol{x} - \boldsymbol{z}\|_2 < c\|\boldsymbol{x}\|_2$.

6. **(a)** Prove that if matrix $A = M - N$ is singular and M is nonsingular, then $\|M^{-1}N\| \geq 1$ where $\|\cdot\|$ is any induced matrix norm.

(b) Prove that if matrix $A = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$ is positive definite, then the Jacobi iteration method converges for a linear system $A\boldsymbol{x} = \boldsymbol{b}$. (Hint: Consider the eigenvalues of the Jacobi iteration matrix $D^{-1}(D - A)$.)

7. Consider the two-point boundary-value problem $y''(x) - p(x)y'(x) - q(x)y(x) = r(x)$, $0 < x < 1$, with $y(0) = y(1) = 0$. Assume that $q(x) \geq \alpha > 0$ for $0 \leq x \leq 1$. Consider the difference scheme

$$\frac{(y_{j+1} - 2y_j + y_{j-1})}{h^2} - p(x_j)\frac{(y_{j+1} - y_{j-1})}{2h} - q(x_j)y_j = r(x_j), \quad \text{for } j = 1, 2, \cdots, N-1,$$

with $y_0 = y_N = 0$, $x_j = jh$, and $h = 1/N$.

(a) Determine the matrix A so that the above difference equations can be written as the linear system $A\boldsymbol{y} = h^2\boldsymbol{r}$ with $\boldsymbol{y} = [y_1, y_2, \cdots, y_{N-1}]^T$ and $\boldsymbol{r} = [r(x_1), r(x_2), \cdots, r(x_{N-1})]^T$.

(b) Prove that if $\frac{h}{2} \max_{0 \leq x \leq 1} |p(x)| \leq 1$, then the $(N-1) \times (N-1)$ matrix A is strictly diagonally dominant.

8. Consider the two-dimensional quadrature formula $\int_{-1}^{1}\int_{-1}^{1} f(x,y)dxdy \approx f(\alpha,\alpha) + f(-\alpha,\alpha) + f(\alpha,-\alpha) + f(-\alpha,-\alpha)$.

Find the value of α such that the formula is exact for every polynomial $f(x,y)$ of degree less than or equal to 3, that is, for $f(x,y) = \sum_{i,j=0}^{3} a_{i,j}x^i y^j$.

9. Consider numerical solution of the initial-value problem $y'(t) = f(t, y(t))$, $0 < t < 1$, $y(0) = y_0 = 0$ using the trapezoidal method $y_{k+1} = y_k + \frac{h}{2}[f(t_k, y_k) + f(t_{k+1}, y_{k+1})]$, for $k = 0, 1, \cdots, N-1$, where $N = 1/h$ and $t_k = kh$. Suppose that $\max_{0 \leq t \leq 1} |y'''(t)| \leq M$ and that $|f(t,z) - f(t,\tilde{z})| \leq L|z - \tilde{z}|$ for all $z, \tilde{z} \in R$. Assuming that $hL < 1$, prove that $\max_{0 \leq k \leq N} |y_k - y(t_k)| \leq ch^2$ where the constant c does not depend on h.

(Note that $\int_{t_k}^{t_{k+1}} g(z)dz - \frac{h}{2}(g(t_{k+1}) + g(t_k)) = -\frac{h^3}{12}g''(\xi_k)$ for some $\xi_k \in (t_k, t_{k+1})$.)