Numerical Analysis Preliminary Examination 2002

Department of Mathematics and Statistics

Note: Do **eight** of the following nine problems. **Clearly indicate which eight are to be graded**.

1. Suppose that $f \in C[a,b]$ has a unique zero $x^* \in [a,b]$, $f(a) < 0$ and $f(b) > 0$. Define the two sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ by $x_0 = a$ and $y_0 = b$ and for $n = 1, 2, 3, \ldots$

   (i) if $f\left(\dfrac{x_{n-1}+y_{n-1}}{2}\right) < 0$ then $x_n = \dfrac{x_{n-1}+y_{n-1}}{2}$ and $y_n = y_{n-1}$

   (ii) if $f\left(\dfrac{x_{n-1}+y_{n-1}}{2}\right) \geq 0$ then $x_n = x_{n-1}$ and $y_n = \dfrac{x_{n-1}+y_{n-1}}{2}$

   Prove that

   (a) $x^* \in [x_n, y_n]$, $f(x_n) < 0$ and $f(y_n) \geq 0$ for $n = 0, 1, 2, \ldots$.

   (b) $|x_n - x^*| \leq \dfrac{b-a}{2^n}$ for $n = 0, 1, 2, \ldots$.

2. Let $n \times n$ matrix U be a non-singular upper triangular matrix with elements u_{ij}. Consider the linear system $U\vec{x} = \vec{b}$. Describe an efficient algorithm for calculating $\vec{x}$. Prove that your method only requires n^2 arithmetic operations.

3. Consider interpolating the function $f(x,y)$ at the n^2 points (x_i, y_j) for $i, j = 1, 2, \ldots, n$ where $\{x_i\}_{i=1}^{n}$ and $\{y_j\}_{j=1}^{n}$ are each pairwise distinct. Let $l_i(x) = \prod_{\substack{m=1 \\ m \neq i}}^{n} \dfrac{x - x_m}{x_i - x_m}$ and $\hat{l}_j(y) = \prod_{\substack{k=1 \\ k \neq j}}^{n} \dfrac{y - y_k}{y_j - y_k}$. Let $p(x,y) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} l_i(x) \hat{l}_j(y)$.

   (a) Find c_{ij} for $i, j = 1, \ldots, n$ so that $p(x,y)$ interpolates $f(x,y)$ at the n^2 points.

   (b) Show that $\sum_{i=1}^{n}\sum_{j=1}^{n} l_i(x)\hat{l}_j(y) = 1$.

4. Consider the initial-value problem $\dfrac{dy}{dt} = f(y,t)$, $y(0) = y_0$, for $0 \leq t \leq 1$. Suppose that $|f(y,t) - f(z,t)| \leq L|y - z|$ for $0 \leq t \leq 1$ and $y, z \in \mathbb{R}$. Also, suppose that the solution $y(t)$ satisfies $\max_{0 \leq t \leq 1} |y''(t)| = M$. Consider the numerical scheme $y_{n+1} = y_n + hf(x_n, t_n) + \epsilon_n$ for $n = 0, 1, 2, \ldots, N-1$ where $t_n = nh$, $h = 1/N$ and $y_n \approx y(t_n)$. The ϵ_n are rounding errors and $|\epsilon_n| < \delta$ for all n. Prove that there are constants $c_1, c_2 > 0$ such that, $|y(1) - y_N| < c_1 h + c_2 \dfrac{\delta}{h}$.

5. Approximate the circular quarter arc γ given by $y(t) = \sqrt{1-t^2}, 0 \le t \le 1$, by a straight line $l(t)$ in the least squares sense using the weight function $w(t) = (1-t^2)^{-1/2},\ 0 \le t \le 1$. (Recall that $< f, g >= \int_0^1 f(t)g(t)w(t)\ dt$).

6. Let A be an invertible matrix. Suppose $A, \Delta A \in \mathbb{R}^{n\times n}$ and $b, \Delta b, x, y \in \mathbb{R}^n$ such that $Ax = b$ and $(A + \Delta A)y = b + \Delta b$. Further let $\delta > 0$ be such that

$$||\Delta A|| \le \delta\ ||A||, \qquad ||\Delta b|| \le \delta\ ||b||, \qquad \delta\ \mathcal{K}(A) = r < 1$$

where $\mathcal{K}(A) = ||A||||A^{-1}||$ is the condition number of the matrix A.

(a) Show that $A + \Delta A$ is non-singular.

(b) Prove that $\dfrac{||y||}{||x||} \le \dfrac{1+r}{1-r}$.

7. (a) Suppose the function $f(x) = \ln(2 + x),\ -1 \le x \le 1$, is interpolated by a polynomial P_n of degree $\le\ n$ at the Chebyshev points $x_k = \cos\left(\dfrac{2k+1}{2n+2}\pi\right)$ for $k = 0, 1, \ldots, n$. Derive a bound for the maximum error $||f - P_n||_\infty = \max\limits_{-1\le x\le 1} |f(x) - P_n(x)|$.

(b) Compare the result of part (a) with a bound for $||f - t_n||_\infty$, where $t_n(x) = \sum_{k=0}^{n} \dfrac{f^{(k)}(0)}{k!}x^k$ is the n^{th} degree Taylor polynomial of f. Also, compare the bounds in parts (a) and (b) for large n.

8. Consider the linear system $A\vec{x} = \vec{b}$ where $A = L + D + U$ and L is strictly lower triangular, D is diagonal, and U is strictly upper triangular. The SOR iterative method has the form $\vec{x}^{(k+1)} = T_\sigma \vec{x}^{(k)} + \vec{c}$ where $\vec{c} = \left(L + \frac{1}{\sigma}D\right)^{-1}\vec{b}$ and $T_\sigma = (\sigma\ L + D)^{-1}\left[(1-\sigma)D - \sigma U\right]$. Let $A = \begin{bmatrix} 2 & -5 \\ 1 & 2 \end{bmatrix}$ and $\vec{x}^{(0)} = \vec{b} = [1,\ 1]^T$. Prove that the SOR method with $\sigma = 1$ is not convergent but the SOR method with $\sigma = \frac{1}{2}$ is convergent.

9. Let $f \in C^2[a, b]$. It is known that

$$\int_a^b f(x)\ dx - f\left(\frac{a+b}{2}\right)(b-a) = \frac{(b-a)^3}{24} f''(\xi)$$

for some $\xi \in [a, b]$. Prove that if $f \in C^2[0, 1]$, then

$$\left| \int_0^1 f(x)\ dx - \sum_{k=0}^{N} \frac{1}{N} f\left(\frac{k}{N} + \frac{1}{2N}\right) \right| \le \frac{M}{24N^2}$$

where $M = \max\limits_{0\le x\le 1} |f''(x)|$.