Numerical Analysis Preliminary Examination 2002

Department of Mathematics and Statistics

Note: Do **eight** of the following nine problems. **Clearly indicate which eight are to be graded**.

1. Let A be a real, symmetric, strictly diagonally dominant $n \times n$ matrix. Suppose $A = D + N$, where D is a diagonal matrix and $N_{ii} = 0$ for each i.

   (a) Show that $A\vec{x} = \vec{b}$ if and only if $\vec{x} = D^{-1}(b - N\vec{x})$.

   (b) Show that there exists a $\rho < 1$ such that for $f(\vec{x}) = D^{-1}(b - N\vec{x})$,
   $$||f(\vec{x}) - f(\vec{y})||_\infty \le \rho \; ||\vec{x} - \vec{y}||_\infty.$$

   (c) Show that the sequence $\vec{x}^{(k+1)} = D^{-1}(b - N\vec{x}^{(k)})$ converges to $\vec{x}$.

2. Let the $n \times n$ matrix A have elements
   $$a_{ij} = \int_0^1 e^{ix} \, e^{jx} \, dx$$
   for $1 \le i, j \le n$. Prove that A has a Cholesky Factorization $A = L^T L$.

3. Let A be a nonsingular $n \times n$ real matrix and $||A^{-1}B|| = r < 1$.

   (a) Show that $A + B$ is nonsingular and $||(A+B)^{-1}|| \le \dfrac{||A^{-1}||}{1-r}$.

   (b) Show that
   $$||(A+B)^{-1} - A^{-1}|| \le \frac{||B|| \; ||A^{-1}||^2}{1-r}.$$

4. Let x_i^* for $i = 1, 2, \cdots, n$ be positive numbers on a computer. With a unit round-off error δ, $x_i^* = x_i \; (1 + \epsilon_i)$ with $|\epsilon_i| \le \delta$, where x_i for $i = 1, 2, \cdots, n$ are the exact numbers.

   (a) Consider the product $P_n = \prod_{i=0}^{n} x_i$ and its floating point approximation $P_n^* = \prod_{i=0}^{n} x_i^*$. Show that if $P_n^* = P_n(1 + \epsilon)$, then ϵ satisfies
   $$\epsilon \le e^{\delta(2n+1)} - 1.$$

   (b) Consider the scalar product $S_2 = \vec{a}^T \, \vec{b}$ where $\vec{a} = (x_1, x_2)^T$ and $\vec{b} = (x_3, x_4)^T$. Let S_2^* be the floating point approximation of S_2. Prove that
   $$\frac{S_2^*}{S_2} \le e^{4\delta}.$$

5. Assume that $f \in C^3[a,b]$ and $x_0, \; x_0 + h, \; x_0 + 2h \in [a,b]$. Prove that there exist constants c_1 and c_2 such that
   $$\left| f'(x_0) - \frac{1}{h}\left[-\frac{3}{2} f(x_0) + c_1 f(x_0 + h) + c_2 f(x_0 + 2h) \right] \right| \le c \; h^2 \max_{a \le x \le b} |f'''(x)|$$
   where $c > 0$ is a constant independent of h.

6. Let $f(x) = \frac{1}{x}$ and $P_2(x)$ be the Lagrange quadratic polynomial that interpolates $f(x)$ at $x_0 = 2,\ x_1 = 2.5$ and $x_2 = 4$. Recall the error formula

$$f(x) - P_2(x) = \frac{1}{6}\ (x - x_0)\ (x - x_1)\ (x - x_2) f'''(\xi(x)), \quad x_0 < x < x_2\ .$$

   (a) Using the error formula, obtain a sharp error bound for $|f(3) - P_2(3)|$.

   (b) Find a function $\xi(x)$ explicitly for this problem.

7. Consider a quadrature formula of the type

$$\int_0^\infty e^{-x} f(x)\ dx = af(0) + bf(c) + E(f)$$

where $E(f)$ is the error in the formula.

   (a) Find a, b and c such that the formula is exact for polynomials of the highest degree possible. (Note that $\int_0^\infty e^{-x}\ x^n\ dx = n!$).

   (b) Let $P(x)$ be the Hermite polynomial interpolating f at the (simple) point $x = 0$ and double point $x = 2$; i.e. $P(0) = f(0)$, $P(2) = f(2)$ and $P'(2) = f'(2)$. Determine $\int_0^\infty e^{-x}\ P(x)\ dx$ and compare with the result in part (a).

   (c) For the values of a, b and c found in part(a), obtain the error $E(f)$ in the form $E(f) = Cf'''(\xi)$ for some $\xi > 0$ where C is a constant.

8. Consider the initial-value problem $y'(t) = f(t, y)$ for $0 \leq t \leq 1$ with $y(0) = a$. Consider the one-step method

$$y_{k+1} = y_k + h\ \Phi(t_k, y_k, h)$$

with $y_0 = a,\ h = 1/N$, and $t_k = kh$ for $k = 0, 1, \cdots, N$. Assume that there is a constant L such that

$$|\Phi(t, y, h) - \Phi(t, z, h)| \leq L\ |y - z|$$

for all $t, y, z \in \mathbb{R}$. Furthermore, assume that the solution $y(t)$ satisfies

$$|y(t + h) - y(t) - h\Phi(t, y(t), h)| \leq c\ h^{p+1}$$

for all $t, h \in [0, 1]$. Prove that

$$|y_N - y(1)| \leq c\ \frac{h^p}{L}\ (e^L - 1).$$

9. Let $f : [a, b] \to \mathbb{R}$ be a C^1 function satisfying $f'(x) \neq 0$ for $x \in [a, b]$. Let $\{p_n\}_{n=0}^\infty$ be the Newton iteration sequence for solving $f(x) = 0$ i.e., p_n satisfies

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}.$$

Assume that $p_n \in (a, b)$ for $n \geq 0$ and $\lim_{n\to\infty} p_n = r$.

   (a) Show that

$$f(r) = 0.$$

   (b) Prove that

$$|p_n - r| \leq \max_{x\in[a,b]} \frac{|f(p_n)|}{|f'(x)|}.$$