Numerical Analysis Preliminary Examination August 2003

Department of Mathematics and Statistics

Note: Do **eight** of the following ten problems. **Clearly indicate which eight are to be graded**.

1. Suppose that $f \in C^1[0,1]$ satisfies

$$f(x) = \int_0^1 f(y)g(y,x)\, dy + b(x)$$

where $g \in C^1([0,1]\times[0,1])$, $b \in C^1[0,1]$, and $\max\limits_{0\le x,y\le 1} |g(y,x)| = \lambda < 1$. Consider the approximation

$$f_k = \sum_{m=0}^{N-1} f_m g(y_m, x_k)\Delta y + b(x_k) \quad \text{for } k = 0, 1, \cdots, N$$

where $\Delta y = \frac{1}{N}$, $x_k = k\Delta y$, and $y_m = m\Delta y$. (Assume that the linear system has a unique solution.) Prove that

$$\max_{0\le k\le N} |f(x_k) - f_k| \le \frac{1}{1-\lambda}\frac{\Delta y}{2} \max_{0\le x,y\le 1} \left| \frac{\partial}{\partial y}(f(y)g(y,x)) \right|$$

(Hint: Use the fact that $\left| \int_0^1 r(y)\, dy - \sum_{m=0}^{N-1} r(y_m)\Delta y \right| \le \frac{\Delta y}{2} \max\limits_{0\le y\le 1} |r'(y)|$.)

2. (a) Let T be a $n\times n$ matrix with $||T||_\infty < 1$. Show that $||(I-T)^{-1}||_\infty \ge \dfrac{1}{1+||T||_\infty}$.

   (b) Let A be $n\times n$ tridiagonal matrix with elements, $a_{ij} = \begin{cases} 5 & \text{if } i=j \\ 1 & \text{if } i=j+1 \text{ or } i=j-1 \\ 0 & \text{otherwise.} \end{cases}$
   Write $A = 5(I-T)$ and show that the spectral radius $\rho(T) < 1$. Moreover show that $||A^{-1}||_\infty \ge \frac{1}{7}$.

3. Consider the singular value decomposition of the matrix $A \in \mathbb{R}^{m\times n}$ given by $A = PDQ$, where P is an $m\times m$ unitary matrix, D is an $m\times n$ diagonal matrix and Q is an $n\times n$ unitary matrix. Show that $||A(A^TA)^{-1}A^T||_2 = 1$. Assume that A^TA is nonsingular. (Note that D is not a square matrix.)

4. Suppose that $y'''(t) = t + 2ty''(t) + 2t^2y(t)$, for $1 \le t \le 2$, $y(1) = 1$, $y'(1) = 2$, $y''(1) = 3$. Convert this third-order problem into a first-order system and compute $\vec{y}_k$ for $k = 1, 2$ for Euler's method with step length $h = 0.1$.

5. Let $||f|| = \left(\int_0^1 e^{-x}(f(x))^2\, dx \right)^{\frac{1}{2}}$ be a norm on $C[0,1]$. Let $g(x) = x$. Find a and b such that if $y(x) = a + be^x$ then $||g(x) - y(x)|| \le ||g(x) - c - de^x||$ for any $c, d \in \mathbb{R}$.

6. Consider solving the initial value problem $y' = \lambda y,\ y(0) = \alpha$ where $\lambda < 0$ by the Implicit Trapezoid Method which is given by

$$w_0 = \alpha, \quad w_{i+1} = w_i + \frac{h}{2}\left[f(t_{i+1}, w_{i+1}) + f(t_i, w_i)\right], \ 0 \le i \le N-1, \ t_i = ih, \ h = \frac{T}{N}.$$

Prove that any two numerical solutions w_i and $\hat{w}_i$ satisfy for $0 \le t_i \le T$

$$|w_i - \hat{w}_i| \le e^K \, |w_0 - \hat{w}_0|$$

where $K = \dfrac{\lambda T}{2}$ and $w_0, \hat{w}_0$ are respective initial values with $w_0 \ne \hat{w}_0$. (That is, w_i and $\hat{w}_i$ satisfy the same difference equations except for different initial values.)

7. Consider the quadrature formula of the type

$$\int_0^1 f(x)\,[x \ \ln(1/x)] \ dx = a_0 f(0) + a_1 f(1).$$

(a) Find a_0 and a_1 such that the formula is exact for linear polynomials. (Note: $\displaystyle\int_0^1 x^n \ln(1/x) \, dx = \left(\frac{1}{n+1}\right)^2$ for $n \ge 0$.)

(b) Describe how the above formula, for $h > 0$, can be used to approximate $\displaystyle\int_0^h g(t) \ t \ \ln(h/t) \ dt$.

8. Suppose that $I(h)$ is an approximation to $\displaystyle\int_a^b f(x) \ dx$ where h is the width of a uniform subdivision of $[a, b]$. Suppose that the error satisfies

$$I(h) - \int_a^b f(x) \ dx = c_1 h + c_2 h^2 + O(h^3)$$

where c_1 and c_2 are constants independent of h. Let $I(h), I(h/2)$, and $I(h/3)$ be calculated for a given value of h. Use the values $I(h), I(h/2)$ and $I(h/3)$ to find an $O(h^3)$ approximation to $\displaystyle\int_a^b f(x) \ dx$.

9. Let $B = I - AC$ be an $n \times n$ matrix with $||B||_\infty \le q < 1$ and A nonsingular. Define the sequence $\{X_k\}_{k=0}^\infty$ of matrices by $X_{k+1} = X_k B + C, \ k = 0, 1, 2, \ldots$ where $X_0 = 0$. Prove that $||X_k - A^{-1}||_\infty \to 0$ as $k \to \infty$.

10. Carefully describe how you would efficiently and accurately compute the value of x that satisfies $\displaystyle\int_0^x e^{t^2} \ dt = 1$.