Numerical Analysis Preliminary Examination May 2003

Department of Mathematics and Statistics

Note: Do **eight** of the following ten problems. **Clearly indicate which eight are to be graded**. No calculators are allowed.

1. Consider an iteration function $g(x)$ of the form $g(x) = x - f(x)f'(x)$. (Note: This is NOT a Newton iteration function.) Assume that r satisfies $f(r) = 0$ and $f'(r) \neq 0$. Find the precise conditions on the function f so that the iterations $x_{n+1} = g(x_n)$ converge to the fixed-point r at least *cubically* if started near r.

2. Consider the problem of finding the least squares polynomial approximation to the function $f(x)$ on $[0, 1]$, i.e., find $p(x) = \sum_{j=1}^{n} \alpha_j x^{j-1}$ such that $\int_0^1 \left(f(x) - \sum_{j=1}^{n} \alpha_j x^{j-1} \right)^2 dx$ is minimized for $\alpha_0, \alpha_1, \cdots, \alpha_n$.

   (a) Find an $n \times n$ matrix A and a vector $\vec{b}$ so that $A\vec{\alpha} = \vec{b}$ where $(\vec{\alpha})_i = \alpha_i$.

   (b) Explain why this is not a good approach for computing $\alpha_1, \alpha_2, \cdots, \alpha_n$.

3. (a) Consider the formula $\int_0^h f(x)\ dx = h\left\{ Af(0) + Bf\left(\frac{h}{3}\right) + Cf(h) \right\}$. Find A, B, C such that this is exact for all polynomials of degree less than or equal to 2.

   (b) Suppose that the Trapezoidal rule applied to $\int_0^2 f(x)\ dx$ gives the value $\frac{1}{2}$ while the quadrature rule in part (a) applied to $\int_0^2 f(x)\ dx$ gives the value $\frac{1}{4}$. If $f(0) = 3$, then show that $f\left(\frac{2}{3}\right) = 1$.

4. Let $s_1(x) = 1 + c(x + 1)^3$, $-1 \leq x \leq 0$, where c is a (real) parameter. Determine $s_2(x)$ on $0 \leq x \leq 1$ so that,

$$s(x) = \begin{cases} s_1(x), & \text{if} \quad -1 \leq x \leq 0 \\ s_2(x), & \text{if} \quad 0 \leq x \leq 1 \end{cases}$$

   is a natural cubic spline, i.e., $s''(-1) = s''(1) = 0$ on $[-1, 1]$ with nodal points at $-1, 0, 1$. How must c be chosen if one wants $s(1) = -1$?

5. Consider the iterative procedure $y_{j+1}^{(k+1)} = y_j + \frac{h}{2}\left[f(y_j) + f(y_{j+1}^{(k)}) \right]$ for $k = 0, 1, 2, \ldots$ where $y_j \in \mathbb{R}$ is given, $f \in C(\mathbb{R})$, and $y_{j+1}^{(0)} = y_j$. Suppose that $|f(u) - f(v)| \leq L|u - v|$ for all $u, v \in \mathbb{R}$ for a constant L. Prove that if $\frac{hL}{2} < 1$, then $y_{j+1}^{(k)} \to y_{j+1}$ as $k \to \infty$ where y_{j+1} satisfies $y_{j+1} = y_j + \frac{h}{2}\left[f(y_j) + f(y_{j+1}) \right]$.

6. Consider the boundary-value problem $y''(x)+y'(x)+xy(x) = \cos(x), 0 < x < 1$, with $y(0) = 1, y(1) = 2$.

   (a) Show how the solutions of two initial-value problems

   $$\begin{aligned} u''(x) + u'(x) + xu(x) &= \cos(x), 0 < x < 1, u(0) = 1, u'(0) = 0 \\ w''(x) + w'(x) + xw(x) &= \cos(x), 0 < x < 1, w(0) = 1, w'(0) = 1 \end{aligned}$$

   can be combined to find $y(x)$. In particular, find a and b such that $y(x) = au(x) + bw(x)$.

   (b) Describe a numerical procedure for approximating $y(x)$ using part (a).

7. Let $A = D-U-L$ where A is strictly diagonally dominant, D is diagonal, U is upper triangular and L is lower triangular. Furthermore, assume that D, U and L have all nonnegative elements, that is, $D, U, L \geq 0$. Suppose $\vec{b} \geq \vec{0}$. Consider the Gauss-Seidel iterative procedure $\vec{x}^{(m+1)} = (D-L)^{-1}U\vec{x}^{(m)} + (D-L)^{-1}\vec{b}$ for $m = 0, 1, 2, \ldots$ with $\vec{x}^{(0)} = \vec{0}$. Assume also that the spectral radius $\rho((D-L)^{-1}U) = \gamma < 1$. Prove that $\vec{x}^{(m)} \to \vec{x}$ where all the elements of $\vec{x}$ are nonnegative, that is, $\vec{x} \geq \vec{0}$. (Hint: Show that $\vec{x}^{(m)} \geq 0$ for each m.)

8. The polynomial $P_n(x)$ interpolating the function $f(x)$ at the nodes x_k for $k = 0, \ldots, n$ is given by, $P_n(x) = \displaystyle\sum_{k=0}^{n} L_k(x)\ f(x_k)$, where $L_k(x) = \displaystyle\prod_{i=0, i\neq k}^{n} \frac{(x-x_i)}{(x_k - x_i)}$. Let us define $\psi(x) = \displaystyle\prod_{i=0}^{n}(x-x_i)$ and $\lambda_k^{(n)} = \displaystyle\prod_{j=0, j\neq k}^{n} \frac{1}{(x_k - x_j)}$.

   (a) Show that $\displaystyle\sum_{k=0}^{n} L_k(x) = 1$ and $P_n(x) = \psi(x)\displaystyle\sum_{k=0}^{n} \frac{f(x_k)}{(x-x_k)\psi'(x_k)}$.

   (b) Show that if x is not a node, then $P_n(x) = \dfrac{\displaystyle\sum_{k=0}^{n} f(x_k)\lambda_k^{(n)}(x-x_k)^{-1}}{\displaystyle\sum_{m=0}^{n} \lambda_m^{(n)}(x-x_m)^{-1}}$.

9. Let $f \in C^1([0,1] \times [0,1])$ and $h = \dfrac{1}{n}$. Show that

   $$\left| \int_0^1 \int_0^1 f(x,y)\ dxdy - h^2 \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(ih, jh) \right| \leq ch$$

   for some constant c independent of h. (Recall $f(x,y) = f(a,b) + \frac{\partial f}{\partial x}(\mu,\xi)(x-a) + \frac{\partial f}{\partial y}(\mu,\xi)(y-b)$ for some μ between x and a and some ξ between y and b.)

10. Suppose $||I - AB_0|| = c < 1$ and $B_k = B_{k-1} + B_{k-1}(I - AB_{k-1}),\ k = 1, 2, \ldots$.

   (a) Show that $||I - AB_k|| \leq c^{2^k}$.

   (b) Suppose A is nonsingular, show that $||A^{-1} - B_k|| \leq ||A^{-1}||\ c^{2^k}$.