Numerical Analysis Preliminary Examination August 2004

Department of Mathematics and Statistics

Note: Do **eight** of the following ten problems. **Clearly indicate which eight are to be graded**. No calculators are allowed.

1. Consider the following algorithm.

$x_0 = 0$

$x_1 = 1$

error $=$ abs$(x_1 - x_0)$

eps $= .00001$

while (error $>$ eps) do the following steps

$f_0 = \exp(x_0) + x_0 - 2$

$f_1 = \exp(x_1) + x_1 - 2$

$x_2 = x_1 - (x_1 - x_0) * f_1/(f_1 - f_0)$

$x_0 = x_1$

$x_1 = x_2$

error $=$ abs$(x_1 - x_0)$

end of while

   (a) Clearly describe the purpose of the algorithm. (It is not necessary to perform calculations.)

   (b) Change the numerical method and the algorithm so that it is quadratically convergent.

2. Consider the matrix $A = \begin{bmatrix} -.4 & 1.0 & -.08 \\ 1.2 & -2.0 & 1.4 \\ -.6 & 1.0 & -.2 \end{bmatrix}$ with inverse $A^{-1} = \begin{bmatrix} 5.0 & 3.0 & 1.0 \\ 3.0 & 2.0 & 2.0 \\ 0.0 & 1.0 & 2.0 \end{bmatrix}$

   (a) Compute $\|A\|_1$. What is the condition number of A in the 1-norm ?

   (b) Suppose $A\vec{x} = \vec{b}$ and $(A+E)\vec{y} = \vec{b}$ with $\|E\|_1 \leq .01$. Compute the bound for $\dfrac{\|\vec{x}-\vec{y}\|_1}{\|\vec{x}\|_1}$?

3. Consider the iteration method $x_{k+1} = \phi(x_k), k = 0, 1, \ldots$ for solving the nonlinear equation $f(x) = 0$. Consider choosing an iteration function of the form

$$\phi(x) = x - af(x) - b(f(x))^2 - c(f(x))^3$$

   where a, b, c are parameters to be determined. Find expressions for the parameters a, b, c such that the iteration method is of fourth order.

4. Let A be a $n \times n$ matrix such that $A^k \to 0$ as $k \to \infty$. Then show that,

   (a) $I - A$ is nonsingular.

   (b) $||(I-A)^{-1}|| \leq \dfrac{1}{1-||A||}$ (Hint: First show that $(I-A)^{-1} = \displaystyle\sum_{k=0}^{\infty} A^k$.)

   (c) Consider the iteration $\vec{x}^{(k)} = A\vec{x}^{(k-1)} + \vec{c}$, where $\vec{c}$ is a given vector. Find $\vec{z}$ in terms of A and $\vec{c}$ such that $||\vec{x}^{(k)} - \vec{z}|| \to \vec{0}$ as $k \to \infty$.

5. Given the following differential equation $\dfrac{dy}{dx} = f(x, y)$

   (a) Define the truncation error for the following two-step method
   $$y_{n+1} = a_0 y_n + a_1 y_{n-1} + h[b_{-1} f(x_{n+1}, y_{n+1}) + b_0 f(x_n, y_n) + b_1 f(x_{n-1}, y_{n-1})] .$$

   (b) Find conditions on the coefficients a_0, a_1, b_{-1}, b_0, b_1 to make it a third order method.

6. Let $\{\vec{v}^{(1)}, \vec{v}^{(2)}, \cdots, \vec{v}^{(n)}\}$ be a set of nonzero vectors associated with a positive definite matrix A which satisfy $\langle \vec{v}^{(i)}, A\vec{v}^{(j)} \rangle = 0$ if $i \neq j$, where the innerproduct is given by $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y}$.

   (a) Show that the given set of nonzero vectors associated with A is linearly independent.

   (b) If $A\vec{x} = \vec{b}$, find $\vec{x}$ in terms of $\vec{v}^{(1)}, \vec{v}^{(2)}, \cdots, \vec{v}^{(n)}$.

7. Let $f \in C^2[a, b]$. It is known that $\displaystyle\int_a^b f(x)dx - \frac{b-a}{2}[f(a) + f(b)] = -(b-a)^3 \frac{f''(\xi)}{12}$ for some $\xi \in [a, b]$. Prove that for $f \in C^2[0, 1]$ and $M = \max\limits_{0 \le x \le 1} |f''(x)|$,
   $$\left| \int_0^1 f(x)dx - \sum_{k=0}^{N-1} \frac{1}{2N} \left[f\left(\frac{k}{N}\right) + f\left(\frac{k+1}{N}\right) \right] \right| \le \frac{M}{12N^2} .$$

8. Let A be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_n 0$ where $\lambda_1 = 1$, $\lambda_2 = \frac{1}{2}$ along with the corresponding eigenvectors $\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_n$ where $\|\vec{z}_i\|_\infty = 1$ for each i.

   (a) Let $\vec{x}_k = A\vec{x}_{k-1}$ for $k = 1, 2, \ldots$ where $\vec{x}_0 = \vec{z}_1 + 8\vec{z}_2$. Find the number of iterations k so that $\|\vec{x}_k - \vec{z}_1\|_\infty \le \dfrac{1}{2^{17}}$.

   (b) Now consider $\vec{x}_k = (10A - 9I)^{-1}\vec{x}_{k-1}$ for $k = 1, 2, \ldots$ where $\vec{x}_0 = \vec{z}_1 + 8\vec{z}_2$. Find the number of iterations k so that $\|\vec{x}_k - \vec{z}_1\|_\infty \le \dfrac{1}{2^{17}}$.

9. Let $f \in C^6[-1, 1]$.

   (a) Construct the Hermite interpolating polynomial $p(x)$ on the interval $[-1, 1]$ such that $p(x_i) = f(x_i)$ and $p'(x_i) = f(x_i)$ for $x_i = -1, 0, 1$.

   (b) Give an expression for interpolation error $\text{Err}(f) = p(x) - f(x)$.

   (c) Show that the following quadrature formula
   $$\int_{-1}^1 f(x)dx = \frac{7}{15} f(-1) + \frac{16}{15} f(0) + \frac{7}{15} f(1) + \frac{1}{15} f'(-1) - \frac{1}{15} f'(1)$$
   is exact for all polynomials of degree ≤ 5.

10. Consider the scalar initial-value problem $\dfrac{dy}{dt}(t) = f(y(t)), 0 \le t \le 1, y(0) = y_0$ along with the numerical method $y_{i+1} = y_i + \frac{3}{2} h f(y_i) - \frac{1}{2} h f(y_{i+1})$ for $i = 0, 1, \ldots, N-1$, where $h = 1/N$. Suppose that $|f(u) - f(v)| \le L\,|u - v|$ for $u, v \in \mathbb{R}$ and $\max\limits_{0 \le t \le 1} |y''(t)| \le M$. Prove that if $Lh \le 1$ so that $\dfrac{1 + \frac{3hL}{2}}{1 - \frac{hL}{2}} \le 1 + 4hL$, then $\max\limits_{0 \le i \le N} |y_i - y(t_i)| \le \dfrac{M}{2L} e^{4L} h$ where y_i is the approximate solution and $y(t_i)$ is the exact solution at $t_i = ih, i = 0, 1, \cdots, N-1$.
    (Hint: You may use the Taylor-type formula: $y(t_{i+1}) = y(t_i) + \frac{3h}{2} y'(t_i) - \frac{h}{2} y'(t_{i+1}) + h^2 y''(\xi_i)$ for some $\xi_i \in [t_i, t_{i+1}]$.)