Numerical Analysis Preliminary Examination May 2004

Department of Mathematics and Statistics

Note: Do **eight** of the following ten problems. **Clearly indicate which eight are to be graded**. No calculators are allowed.

1. Consider the following algorithm to estimate $\int_0^1 \int_0^x f(x,y)dydx$. Determine the total number of times that $f(x,y)$ is evaluated.

   $s = 0$

   for $j = 1, 2, \ldots, n$

       for $k = 1, 2, \ldots, j$

           $s = s + f(j/n, k/n)/(n * n)$

       end (k loop)

   end (j loop)

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2. Consider using the iterative refinement procedure $\vec{x}^{(k+1)} = \vec{x}^{(k)} + B(\vec{b} - A\vec{x}^{(k)})$ for $k \geq 0$, to solve $A\vec{x} = \vec{b}$ for $\vec{x} \in \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^n$ with an approximate solution $\vec{x}^{(0)} = B\vec{b}$, where B is an approximate inverse of A. Let $||I - BA|| < 1$.

   (a) Show that the iterative refinement procedure described above produces the sequence of vectors, $\vec{x}^{(m)} = \sum_{k=0}^{m} (I - BA)^k B\, \vec{b}$ for $m \geq 0$.

   (b) Show that $\vec{x}^{(m)}$ converges to $\vec{x}$ as $m \to \infty$.

3. Consider the initial-value system $\frac{d\vec{y}}{dt} = (I - Bt)^{-1}\vec{y}$ for $0 \leq t \leq 1$ where $\vec{y}(t) \in \mathbb{R}^n$, $\vec{y}(0) = \vec{y}_0$, and B is an $n \times n$ matrix with $||B||_\infty \leq \frac{1}{2}$. Euler's method for approximating $\vec{y}(t)$ has the form $\vec{y}_{i+1} = \vec{y}_i + h(I - Bt_i)^{-1}\vec{y}_i = (I + h(I - Bt_i)^{-1})\vec{y}_i$ for $i = 0, 1, \ldots, N-1$, where $t_i = ih$ and $h = \frac{1}{N}$. (Note that $||Bt_i||_\infty \leq \frac{1}{2}$ for all i.)

   (a) Prove that $\|\vec{y}_{i+1}\|_\infty \leq (1 + 2h)\|\vec{y}_i\|_\infty$ for $i = 0, 1, \ldots, N-1$.

   (b) Show that $\|\vec{y}_N\|_\infty \leq e^2\|\vec{y}_0\|_\infty$ for any value of $N \geq 1$.

4. Assume $f(x)$ to be a real function. Let x_0, x_1 be two distinct points.

   (a) Prove that there is a unique polynomial $p(x)$ of degree 3 such that $p(x_j) = f(x_j)$ and $p'(x_j) = f'(x_j)$ for $j = 0, 1$.

   (b) Determine explicitly the polynomial interpolant described in part (a). Also give a formula for the error.

5. Suppose that $f(x)$ satisfies a Lipschitz condition $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in [0, 1]$. Let $\Psi(x)$ be a piecewise constant approximation to $f(x)$ on $[0, 1]$ such that $\Psi(x) = f(x_i)$ for $x_i \leq x < x_{i+1}$ for $i = 0, 1, \ldots, N-1$ with $x_i = ih$ and $h = \frac{1}{N}$. Prove that $\max_{0 \leq x \leq 1} |\Psi(x) - f(x)| \leq ch$ for some constant $c > 0$.

6. Consider the equation $x^3 - x - 1 = 0$ which has a root ξ between 1 and 2.

   (a) Determine a suitable iteration function $T(x)$ such that ξ is a solution of $x = T(x)$ and $T(x)$ is a contraction over $[1, 2]$.

   (b) Find k such that the n^{th} iterate x_n generated by the equation $x_n = T(x_{n-1})$ for $n \geq 1$, satisfies $|x_n - \xi| \leq k^n |x_0 - \xi|$.

7. Let $\{g_1, g_2, \ldots, g_n\}$ be an orthonormal system in an inner-product space E with the associated inner-product $(\cdot, \cdot)$. Let G be the subspace generated by $g_1, g_2, \ldots, g_n$. Let $f \in E$ and $g^* \in G$ satisfy $f - g^* \perp G$.

   (a) Show that g^* is the best approximation to f in G. That is, show that $||f - g^*|| \leq ||f - g||$ for any $g \in G$. Also show that $g^* = \sum_{i=1}^{n} (f, g_i)\, g_i$.

   (b) Let $||f|| = \sqrt{(f, f)}$. Show that $\sum_{i=1}^{n} |(f, g_i)|^2 \leq ||f||^2$.

8. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the $n \times n$ matrix A. Let $D = P^{-1}AP$ be a diagonal matrix for some nonsingular $n \times n$ matrix P.

   (a) Describe Gershgorin's theorem for localizing eigenvalues.

   (b) For any $n \times n$ matrix B, show that the eigenvalues of $(A + B)$ are the same as the eigenvalues of $(D + P^{-1}BP)$.

   (c) Using parts (a) and (b), show that the eigenvalues of $A + B$ lie in the union of the disks $|\lambda - \lambda_i| \leq \kappa_\infty(P)||B||_\infty$, where $\kappa_\infty(P) = ||P||_\infty ||P^{-1}||_\infty$ is the infinity-norm condition number of the matrix P.

9. Let $\vec{p} = [1, 2, \ldots, n]^T \in R^n$ and let A be the $n \times n$ matrix $A = \vec{p}\vec{p}^T$. Consider the power method of the form $\vec{x}_{i+1} = A\vec{x}_i / \|A\vec{x}_i\|_2$ for $i = 0, 1, 2, \ldots$ with $\vec{x}_0 = [1, 1, \ldots, 1]^T$. Suppose $A\vec{x}_0 \neq \vec{0}$. Show that the sequence $\{\vec{x}_i\}_{i=0}^{\infty}$ converges and determine explicitly the vector $\vec{v} \in \mathbb{R}^n$ to which the sequence converges.

10. Given the initial value problem $\frac{dy}{dt} = f(t, y),\ y(a) = \eta$ for the function $y(t)$ over the interval $a \leq t \leq b$. Consider the general two-step method on the discrete point set defined by $t_n = a + nh$ for $n = 0, \ldots, m$ with $h = (b - a)/m$. If we write $y_n = y(t_n)$ and $f_n = f(t_n, y_n)$ the general two-step method becomes

$$\sum_{j=0}^{2} \alpha_j\, y_{n+j} = h \sum_{j=0}^{2} \beta_j f_{n+j}\ .$$

    Assume that $\alpha_2 = 1$ and $\alpha_0 = c$, where c is a parameter.

    (a) Show that, by selecting $\alpha_1, \beta_0, \beta_1$ and β_2 appropriately, the method is third order for $c \neq -1$.

    (b) Show that if $c = -1$, the order of the method is at most 4.

    (c) Show that if $c = -5$, the method can be third order and explicit.