Numerical Analysis Preliminary Examination August 2005

Department of Mathematics and Statistics

Note: Do **nine** of the following ten problems. **Clearly indicate which nine are to be graded.** No calculators are allowed.

1) Consider a piecewise continuous approximation $I_N f(x)$ of the function $f(x)$ over the interval $[0,1]$ with $N+1$ equally spaced points.

   a) Find the largest step $h = 1/N$ for which $f(x) = e^x$ can be approximated with accuracy 10^{-6} by using piecewise continuous linear interpolation.

   b) Find the largest step $h = 1/N$ for which $f(x) = e^x$ can be approximated with accuracy 10^{-6} by using piecewise continuous cubic interpolation.

2) Let $g(x)$ be in $C[-1,1]$. Consider the approximation of the following integral

$$\int_{-1}^{1} \sqrt{1-x^2}\, g(x)\,dx$$

   with 3-point Gaussian formula.

   a) Determine the nodes and the weights of the 3-point Gaussian formula. (The orthogonal polynomials with respect to the weight function $\sqrt{1-x^2}$ can be generated with the recursive formula $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$ for $n = 2, 3, \ldots$ and $U_0(x) = 1$ and $U_1(x) = 2x$.)

   b) Write explicitly the numerical 3-point Gaussian formula for computing the integral.

3) Given the following linear multistep formulas

   (A) $y_{n+1} - \frac{8}{19}y_n + \frac{8}{19}y_{n-1} - y_{n-3} = \frac{6h}{19}(f_{n+1} + 4f_n + 4f_{n-2} + f_{n-3})$, and

   (B) $y_{n+1} + y_n - y_{n-1} - y_{n-2} = 2h(f_n + f_{n-1})$, where $f_n = f(t_n, y_n)$,

   for solving the initial value problem

$$\frac{dy}{dt} = f(t,y) \qquad y(t_0) = y_0$$

   over the interval $t_0 \le t \le T$.

   a) Determine the stability of the schemes (A) and (B).

   b) Find the order of accuracy and the leading error term of (A) and (B).

   c) Discuss consistency and convergence of the schemes (A) and (B).

4) Consider the Runge-Kutta three-stage method

$$y_{i+1} - y_i = \frac{h}{9}(2g_1 + 3g_2 + 4g_3), \qquad g_1 = f(t_i, y_i),\ g_2 = f(t_i + \frac{h}{2}, y_i + \frac{h}{2}g_1),\ g_3 = f(t_i + \frac{3h}{4}, y_i + \frac{3h}{4}g_2)$$

   for solving the initial value problem $\frac{dy}{dt} = f(t,y)$ with $y(0) = y_0$.

   a) Verify that it is a third-order method.

   b) Discuss stability and convergence of the method.

5) Let $A = B - C$, where A, B, C are $n \times n$ nonsingular matrices, and let

$$B\vec{x}^{(m)} = C\vec{x}^{(m-1)} + \vec{y} \qquad m \ge 1.$$

   Show that if $\|B^{-1}C\| < 1$ then $\lim_{m\to\infty} \vec{x}^{(m)} = A^{-1}\vec{y}$ for any $\vec{y}, \vec{x}^{(0)} \in \mathbb{R}^n$.

6) Consider the iteration method

$$x_{k+1} = \phi(x_k) \qquad k = 0, 1, \ldots$$

for solving the equation $f(x) = 0$. Choose the iteration function of the form

$$\phi(x) = x - \gamma_1 f(x) - \gamma_2 f^2(x)$$

and find γ_1 and γ_2 such that the iteration method is at least of the third order. (Suppose that there is a $\xi \in \mathbb{R}$ such that $f(\xi) = 0$, $f'(\xi) \neq 0$, and $f''(\xi) \neq 0$ with $f \in C^2(\mathbb{R})$.)

7) Determine the values of a, b and c so that

$$f(x) = \begin{cases} 3 + x - 9x^2, & x \in [0, 1] \\ a + b(x-1) + c(x-1)^2 + d(x-1)^3, & x \in [1, 2] \end{cases}$$

is a cubic spline having knots at 0, 1 and 2. Determine d so that $\int_0^2 [f''(x)]^2 dx$ is a minimum.

8) Consider the following algorithm

input x_0
$i = 0$
eps $= 1.0$
while (*eps* $> .00001$) *do the following steps*
$x_{i+1} = \frac{1}{3}\cos(x_i) - \frac{1}{2}x_i$
$i = i + 1$
eps $= |x_{i+1} - x_i|$.

a) A standard numerical method is used in this algorithm to generate the sequence $x_0, x_1, x_2, \ldots$. Give the name of this method.

b) Given any $x_0 \in \mathbb{R}$ in the algorithm, prove that *eps* will eventually be less than .00001, that is, prove that the sequence $x_0, x_1, x_2, \ldots$ converges for any $x_0 \in \mathbb{R}$.

9) Consider the midpoint rule $\int_0^h f(x)dx \approx hf(\frac{h}{2})$.

a) Prove that $\left| \int_0^h f(x)dx - hf(\frac{h}{2}) \right| \leq c_1 h^2$ for $f \in C^1[0, h]$ and for a positive constant c_1 independent of h.

b) Prove that $\left| \int_0^h f(x)dx - hf(\frac{h}{2}) \right| \leq c_2 h^3$ for $f \in C^2[0, h]$ and for a positive constant c_2 independent of h.

10) Consider the linear system $A\vec{x} = \vec{b}$ where A is a nonsingular $n \times n$ matrix. Let ΔA be a perturbation of A satisfying $\|\Delta A\| \|A^{-1}\| < 1$. Prove that if $\Delta\vec{x}$ satisfies

$$(A + \Delta A)(\vec{x} + \Delta\vec{x}) = \vec{b},$$

then

$$\frac{\|\Delta\vec{x}\|}{\|\vec{x}\|} \leq \frac{\|A\| \|A^{-1}\|}{1 - \|\Delta A\| \|A^{-1}\|} \frac{\|\Delta A\|}{\|A\|}.$$