

Numerical Analysis Preliminary Examination, August 2006
Department of Mathematics and Statistics

Do nine of the following ten problems. Clearly indicate which nine are to be graded. Calculators are not allowed.

1. Let \vec{x} , with $\|\vec{x}\|_2 = 1$, be an eigenvector of a real symmetric $n \times n$ matrix A , that is, $A\vec{x} = \lambda\vec{x}$ for some $\lambda \in \mathbb{R}$ where $\vec{x}^T\vec{x} = 1$. Let U be an $n \times (n-1)$ matrix such that the $n \times n$ matrix $B = (\vec{x}, U)$ is orthogonal, i.e., $B^T B = (\vec{x}, U)^T (\vec{x}, U) = \begin{bmatrix} \vec{x}^T\vec{x} & \vec{x}^T U \\ U^T \vec{x} & U^T U \end{bmatrix} = I$.

(a) Prove that $B^T A B = \begin{bmatrix} \lambda & \vec{c}^T \\ \vec{0} & C \end{bmatrix}$ where \vec{c} and $\vec{0}$ are vectors of length $n-1$ and C is an $(n-1) \times (n-1)$ matrix.

(b) Prove that C has the same eigenvalues as A except for λ .

2. Prove that $\left| \sin x - \frac{6x}{6+x^2} \right| \leq \frac{x^5}{24}$ for $0 \leq x \leq 2$.

3. Let $g : \mathbb{R}^3 \rightarrow \mathbb{R}$ where $g \in C^2(\mathbb{R}^3)$. Assume that g achieves its minimum at $\vec{x}^* \in D \subset \mathbb{R}^3$ where $D = [-1, 1] \times [-1, 1] \times [-1, 1]$. Let $\vec{x}_0 \in D$ be an initial guess to \vec{x}^* . In the method of steepest descent, let $\vec{x}_1 \in D$ be the next approximation to \vec{x}^* . Carefully explain how \vec{x}_1 is calculated.

4. Let $f \in C[0, 1]$ but $f \notin C^1[0, 1]$. Consider $E_n = \int_0^1 f(x) dx - \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)$. Prove that given $\epsilon > 0$, there is an $N > 0$ such that $|E_n| < \epsilon$ when $n > N$.

5. Let $\vec{x} \in \mathbb{R}^n$ with $\vec{x} = [x_1, x_2, x_3, \dots, x_n]^T$ and $x_1 \neq 0$. Let $\vec{u} = \vec{x} + \sigma \vec{e}_1$ where $\sigma = \text{sign}(x_1) \|\vec{x}\|_2$ and let $\theta = \frac{1}{2} \|\vec{u}\|_2^2$. Finally, let $U = I - \vec{u}\vec{u}^T/\theta$. Prove that $U^2 = I$ and that $U\vec{x} = -\sigma \vec{e}_1$.

6. Let A be a 4×4 matrix with eigenvalues 1, 3, 5, 6 and corresponding eigenvectors $[1, 1, 1, -1]^T$, $[1, 1, 1, 1]^T$, $[1, 1, -1, 1]^T$, and $[1, -1, 1, 1]^T$, respectively. Consider the iteration $\vec{x}^{(k+1)} = A^{-1}\vec{x}^{(k)}$ for $k = 0, 1, 2, \dots$ where $\vec{x}^{(0)} \in \mathbb{R}^4$ is randomly chosen with $\vec{x}^{(0)} \neq \vec{0}$. Prove that $\{\vec{x}^{(k)}\}_{k=0}^\infty$ converges to an $\vec{x}^* \in \mathbb{R}^4$ and find explicitly $\vec{x}^*/\|\vec{x}^*\|_2$.

7. Let $F(\vec{x}) = [f_1(x_1, x_2, \dots, x_n), f_2(x_1, x_2, \dots, x_n), \dots, f_n(x_1, x_2, \dots, x_n)]^T$. Broyden's iterative method of finding $\vec{x} \in \mathbb{R}^n$ so that $F(\vec{x}) = \vec{0}$ has the form for $k = 0, 1, 2, \dots$:

$$\begin{cases} \vec{x}^{(k+1)} = \vec{x}^{(k)} - B_k^{-1} F(\vec{x}^{(k)}) \\ \vec{y}^{(k)} = F(\vec{x}^{(k+1)}) - F(\vec{x}^{(k)}), \quad \vec{s}^{(k)} = \vec{x}^{(k+1)} - \vec{x}^{(k)} \\ B_{k+1} = B_k + \frac{(\vec{y}^{(k)} - B_k \vec{s}^{(k)})(\vec{s}^{(k)})^T}{(\vec{s}^{(k)})^T \vec{s}^{(k)}}. \end{cases}$$

For $n = 1$, prove that Broyden's method reduces to the secant method.

8. Let A be an $n \times n$ matrix with $a_{ij} \leq 0$ if $i \neq j$ and $a_{ii} > 0$ for $1 \leq i, j \leq n$. Let $B = D^{-1}(D - A)$ where D is the diagonal matrix with elements $d_{ij} = 0$ if $i \neq j$ and $d_{ii} = a_{ii}$ for $1 \leq i, j \leq n$. Assume that the spectral radius $\rho(B) < 1$.

(a) Prove that A is nonsingular.

(b) Prove that $A^{-1} \geq 0$, that is, all the entries of A^{-1} are nonnegative.

(c) Let $\vec{c} \in \mathbb{R}^n$ and define an iteration by $\vec{x}^{(k+1)} = B\vec{x}^{(k)} + \vec{c}$ for $k = 0, 1, 2, \dots$ with $\vec{x}^{(0)} = \vec{0}$. Prove that $\{\vec{x}^{(k)}\}_{k=1}^{\infty}$ converges to a vector $\vec{x} \in \mathbb{R}^n$ and find \vec{x} in terms of A , D , and \vec{c} .

9. Let $f \in C[0, 1]$. Consider the problem of finding the least squares fit to $f(x)$ by a polynomial of degree less than or equal to n . That is, find $p^*(x)$ such that

$$\int_0^1 (f(x) - p^*(x))^2 dx \leq \int_0^1 (f(x) - p(x))^2 dx$$

for all polynomials p of degree less than or equal to n .

(a) If $p^*(x) = a_0 + a_1x + \dots + a_nx^n$, show how to determine the values of the coefficients a_0, a_1, \dots, a_n by solving a system of linear equations.

(b) The Legendre polynomials, $L_i(x)$, $i = 0, 1, \dots, n$, are orthogonal on $[0, 1]$ with respect to the weight function $w(x) = 1$. Show how to express $p^*(x)$ in terms of these polynomials.

10. Assume that the initial-value problem

$$\begin{cases} y'(t) = f(t, y) \\ y(a) = A \end{cases}$$

has a unique solution $y(t)$ on the interval $[a, b]$. Consider approximating this solution by the one-step method

$$\begin{cases} y_{k+1} = y_k + h\phi(t_k, y_k) \\ y_0 = \hat{A} \end{cases}$$

for $k = 0, 1, \dots, N - 1$ where $h = (b - a)/N$ and $t_k = a + kh$. Assume that $|y(t + h) - y(t) - h\phi(t, y(t))| \leq ch^{p+1}$ for all $t \in [a, b]$ and that $|\phi(t, u) - \phi(t, v)| \leq L|u - v|$ for all $t \in [a, b]$. Prove that

$$|y(t_m) - y_m| \leq |A - \hat{A}|e^{L(t_m - a)} + \frac{ch^p}{L}(e^{L(t_m - a)} - 1)$$

for any $0 \leq m \leq N$.