# Numerical Analysis Preliminary Examination, May 2006

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Do eight of the following ten problems. Clearly indicate which eight are to be graded. Calculators are not allowed.

**1.** Let A be the $2 \times 2$ matrix $\begin{pmatrix} a & -b \\ -2a & a \end{pmatrix}$ where $a$ and $b$ are real positive numbers.

a) Find all values of $b/a$ such that the Jacobi iteration is convergent.

b) Find all values of $b/a$ such that the Gauss-Seidel iteration is convergent.

**2.** Let $f \in C^{n+2}[0,1]$ and let $p(x) = \sum_{k=0}^{n} f(x_k)L_k(x)$ where $L_k(x) = \displaystyle\prod_{i=0, i\neq k}^{n} \frac{x - x_i}{x_k - x_i}$.

Assume that $x_k = \dfrac{k}{n}$ for $k = 0, 1, 2, \ldots, n$.

Prove that $|p'(0) - f'(0)| \leq \dfrac{1}{(n+1)!} \displaystyle\max_{0 \leq x \leq 1} |f^{(n+1)}(x)| \prod_{k=1}^{n} \frac{k}{n}$ .

**3.** Let $A$ be a symmetric positive definite $n \times n$ matrix. For any $\mathbf{x} \in \mathbb{R}^n$, define $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T A \mathbf{x}}$. Prove that this defines a norm on $\mathbb{R}^n$. (Hint: To show the triangle inequality, use the Cholesky decomposition.)

**4.** Let $f \in C^\infty[0,1]$. Let $S_n$ be the composite trapezoidal rule approximation to $\int_0^1 f(x)\,dx$ with $n$ intervals, that is, $S_n = \frac{1}{2n} \sum_{k=0}^{n-1} \left( f(\frac{k}{n}) + f(\frac{k+1}{n}) \right)$. Let $h = \frac{1}{m}$ and let $S_m, S_{2m}$, and $S_{4m}$ be the values obtained using $n = m, 2m$, and $4m$. Using only the three values $S_m, S_{2m}$, and $S_{4m}$, find an $O(h^6)$ approximation to $\int_0^1 f(x)\,dx$.

**5.** Consider the iterative method $x_{n+1} = F(x_n) = x_n + f(x_n)/g(x_n)$. Assume that the method converges to a point $r$ which is a simple zero of the function $f(x)$ but not a zero of the function $g(x)$. Find $g(r)$ and $g'(r)$ in terms of $f(r)$, $f'(r)$, and $f''(r)$ so that the method has a cubic convergence rate.

**6.** Consider the approximation $\displaystyle\int_{-1}^{1} f(x)\,dx \approx f(a) + f(b)$ where $a$ and $b$ are real numbers.

(a) Find $a$ and $b$ so that the approximation is exact for any cubic polynomial. Prove, for your choice of $a$ and $b$, that the approximation is exact for any cubic polynomial.

(b) Use your approximation to estimate $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ noticing that a change of variables must first be made.

**7.** Let $\vec{f} : \mathbb{R}^n \to \mathbb{R}^n$ and consider the initial-value system $\frac{d}{dt}\vec{y}(t) = \vec{f}(\vec{y}(t))$ with $\vec{y}(0) = \vec{a}$. Assume that there is a constant $\lambda > 0$ such that $\|\vec{f}(\vec{u}) - \vec{f}(\vec{v})\|_\infty \le \lambda\|\vec{u} - \vec{v}\|_\infty$ for all $\vec{u}, \vec{v} \in \mathbb{R}^n$. For approximately solving this initial-value system, consider using the implicit trapezoidal method

$$\vec{y}_{k+1} = \vec{y}_k + \frac{h}{2}\vec{f}(\vec{y}_k) + \frac{h}{2}\vec{f}(\vec{y}_{k+1}) \quad \text{for} \quad k = 0, 1, 2, \ldots, \tag{1}$$

with $\vec{y}_0 = \vec{a}$. To compute $\vec{y}_{k+1}$, the following iterative scheme is employed:

$$\vec{y}_{k+1}^{(m+1)} = \vec{y}_k + \frac{h}{2}\vec{f}(\vec{y}_k) + \frac{h}{2}\vec{f}(\vec{y}_{k+1}^{(m)}) \quad \text{for} \quad m = 0, 1, 2, \ldots, \quad \text{with} \quad \vec{y}_{k+1}^{(0)} = \vec{y}_k.$$

Assume that $\frac{\lambda h}{2} \le \frac{1}{3}$. Prove that $\vec{y}_{k+1}^{(m)} \to \vec{y}_{k+1}$ as $m \to \infty$ where $\vec{y}_{k+1}$ is the solution to (1).

**8.** Let $F : \mathbb{R}^n \to \mathbb{R}^n$ where

$$F(\vec{x}) = [f_1(x_1, x_2, x_3, \ldots, x_n), f_2(x_1, x_2, x_3, \ldots, x_n), \ldots, f_n(x_1, x_2, x_3, \ldots, x_n)]^T.$$

Assume that $f_i \in C^2(\mathbb{R}^n)$ for $i = 1, 2, \ldots, n$. Consider the problem of finding $\vec{x}^* \in \mathbb{R}^n$ such that $F(\vec{x}^*) = \vec{0}$. Describe thoroughly and clearly Newton's method for solving this problem. Explain one difficulty in implementing Newton's method for a large number of equations $n$, say $n = 100$.

**9.** Consider the multistep method for the initial-value problem $y'(t) = f(y(t))$ of the form:

$$y_n + (A-1)y_{n-1} - A\,y_{n-2} = \frac{h}{12}\big((5-A)f(y_n) + 8(1+A)f(y_{n-1}) + (5A-1)f(y_{n-2})\big).$$

For any $A \in \mathbb{R}$, this formula is exact for all cubic polynomial solutions $y(t) = a + bt + ct^2 + dt^3$. That is,

$$y(t_n) + (A-1)y(t_{n-1}) - A\,y(t_{n-2}) = \frac{h}{12}\big((5-A)y'(t_n) + 8(1+A)y'(t_{n-1}) + (5A-1)y'(t_{n-2})\big)$$

for all cubic polynomials where $t_n = nh$ and $h$ is step length.
(a) Determine the value of $A$ so that the method is exact for all polynomials of degree 4.
(b) Analyze stability and consistency of the multistep method using the value of $A$ found in part (a).

**10.** Assume that $n \times n$ matrix $A$ is nonsingular. Show that if $\|AB - I\| = \epsilon < 1$, then

$$\|A^{-1} - B\| \le \|B\| \left(\frac{\epsilon}{1-\epsilon}\right)$$