Numerical Analysis Preliminary Examination August 2007

Department of Mathematics and Statistics

Note: Do **nine** of the following ten problems. **Clearly indicate which nine are to be graded.**

1. Let $f \in C^2(\mathbb{R})$. Consider Newton's method $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, $n \geq 0$ for solving the nonlinear equation $f(x) = 0$. Let $e_n = x_n - r$ where r is a simple zero of f (i.e. $f(r) = 0 \neq f'(r)$).

   (a) Show that $e_{n+1} = \dfrac{1}{2}\dfrac{f''(\xi_n)}{f'(x_n)}e_n^2$, where ξ_n is a number between x_n and r.

   (b) Suppose f is increasing and f is convex (i.e. $f''(x) > 0$) for all $x \in \mathbb{R}$. Show that the Newton iteration will converge to the zero from any starting point.

2. Let $A_\alpha = \alpha D + \alpha L + U$, where D is a $n \times n$ diagonal matrix, L is a $n \times n$ strictly lower triangular matrix, U is a $n \times n$ strictly upper triangular matrix, and $\alpha > 0$ is a positive parameter. Let $A_\alpha \vec{x} = \vec{b}$. Suppose the Gauss-Seidel method converges for $\alpha = 1$.

   (a) Prove that the Gauss-Seidel method converges for any value of $\alpha > 1$.

   (b) Let $A_\alpha = \begin{bmatrix} \alpha & \frac{2}{3} \\ \alpha & \alpha \end{bmatrix}$. Show that the Gauss-Seidel method converges for $\alpha = 1$ but does not converge for $\alpha = 0.5$.

3. Let $P(x)$ be the continuous piecewise linear interpolant to $f(x) = x^3$ on the interval $[0, 10]$ such that $P(k) = f(k)$ for $k = 0, 1, 2, \ldots, 10$.

   (a) Find the exact error $e(x) = |f(x) - P(x)|$ on the interval $[2, 3]$.

   (b) Determine the maximum exact error in $[2, 3]$, i.e., find $\max\limits_{2 \leq x \leq 3} e(x)$.

4. Prove that the eigenvalues of matrix A are unaltered if a row of A is multiplied by a number $c \neq 0$ and the corresponding column is multiplied by $\dfrac{1}{c}$.

5. Let f have derivatives of all orders. For $h > 0$ determine a formula of the form,

$$f'''(x) \approx \frac{1}{h^3}\left[af(x-2h) + bf(x-h) + cf(x) + df(x+h) + ef(x+2h)\right]$$

   where the order of the error is h^2. Find a, b, c, d and e.

6. Consider the initial-value problem $\dfrac{dy}{dt} = a + by(t) + c\sin(y(t))$, $0 \leq t \leq 1$ where $y(0) = 1$ and $a, b, c > 0$ are constants. Let us suppose that the solution satisfies $\max\limits_{0 \leq t \leq 1} |y''(t)| = M < \infty$. Consider the approximation $y_{k+1} = y_k + (a + by_k + c\sin(y_k))h$ for $k = 0, 1, 2, \ldots, N-1$, $y_0 = y(0)$, and $h = \dfrac{1}{N}$. Prove that $|y(1) - y_N| \leq \dfrac{Mhe^{b+c}}{2(b+c)}$.

7. Let $g(\vec{x}) = <\vec{x}, A\vec{x}> -2 <\vec{x}, \vec{b}>$ where A is a positive definite and symmetric $n \times n$ matrix and the inner product is defined as $<\vec{x}, \vec{y}> = \vec{x}^T\vec{y}$.

   (a) Let $\vec{x}$ and $\vec{v}$ be fixed vectors, $\vec{v} \neq \vec{0}$ and let $\hat{t}$ be a real number variable such that $\hat{t} = \dfrac{<\vec{v}, \vec{b} - A\vec{x}>}{<\vec{v}, A\vec{v}>}$. Show that: $g(\vec{x} + \hat{t}\vec{v}) = g(\vec{x}) - \dfrac{<\vec{v}, \vec{b} - A\vec{x}>^2}{<\vec{v}, A\vec{v}>}$.

   (b) Show that if $\vec{\mathbf{x}}^*$ minimizes $g(\vec{x})$, then $\vec{\mathbf{x}}^*$ is a solution to the positive definite linear system $A\vec{\mathbf{x}}^* = \vec{b}$.

8. Consider the following multi-step method:

$$y_{k+1} + \alpha_0 y_k = h\left(\beta_2 f(t_{k+1}, y_{k+1}) + \beta_1 f(t_k, y_k) + \beta_0 f(t_{k-1}, y_{k-1})\right)$$

   for solving the initial-value problem $y'(t) = f(t, y)$.

   (a) Find $\alpha_0, \beta_0, \beta_1, \beta_2$ such that the method is **third** order.

   (b) Is the method **consistent**? If so why? If not why not?

   (c) Is the method **stable**? If so why? If not why not?

9. Let $f \in C[a, b]$ and let $P_n(x) = \sum_{k=0}^{n} a_k x^k$ be a polynomial of degree at most n that minimizes the error $E = \int_a^b (f(x) - P_n(x))^2\, dx$.

   (a) Prove that $\sum_{k=0}^{n} a_k \int_a^b x^{i+k}\, dx = \int_a^b x^i f(x)\, dx$ for $i = 0, 1, \ldots, n$.

   (b) Also show that the system of equations in part (a) has a unique solution.

10. Determine the total number of operations (addition, subtraction, multiplication, division) for the Gaussian Elimination algorithm described below:

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input $n, (a_{ij})$
for $k = 1$ to $n - 1$ do
    for $i = k + 1$ to n do
        $z = \dfrac{a_{ik}}{a_{kk}}$
        for $j = k$ to n do
            $a_{ij} = a_{ij} - z a_{kj}$
        end do
    end do
end do
