Numerical Analysis Preliminary Examination May 2007
Department of Mathematics and Statistics

Note: Do **nine** of the following ten problems. **Clearly indicate which nine are to be graded.**

1. Assume that the equation $x^2 + bx + c = 0$ has two real roots α and β with $|\alpha| < |\beta|$. Show that the iteration method

$$x_{k+1} = \frac{-c}{x_k + b}$$

is convergent to α if x_0 is sufficiently close to α.

2. Let $A \in \mathbb{R}^{(p+q)\times(p+q)}$ be a nonsingular matrix given by $A = \begin{bmatrix} C & B^T \\ B & 0 \end{bmatrix}$ where $C \in \mathbb{R}^{p\times p}$ and $B \in \mathbb{R}^{q\times p}$. Assume A has an LU decomposition with

$$L = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \text{ and } U = \begin{bmatrix} C & Y \\ 0 & Z \end{bmatrix}$$

Find X, Y, Z.

3. Use a two point Gauss quadrature rule to approximate the following integral:

$$I = \int_{-\infty}^{1} \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx\ .$$

Hint: Note that a change of variables is required to transform the limits of integration to -1 and 1.

4. Let $P_N(x)$ be the continuous piecewise linear interpolant to $f \in C[0,1]$ through the points $0 = x_0 < x_1 < x_2 < \ldots < x_N = 1$, where $x_i = ih$ and $h = \frac{1}{N}$. Assume that $|f(z_1) - f(z_2)| \leq L|z_1 - z_2|$ for $z_1, z_2 \in [0,1]$. Prove that

$$\max_{0\leq x\leq 1} |P_N(x) - f(x)| \leq \frac{Lh}{2}\ .$$

Hint: Note that $P_N(x) = f(x_i)\left(\frac{x_{i+1} - x}{h}\right) + f(x_{i+1})\left(\frac{x - x_i}{h}\right)$ for $x_i \leq x \leq x_{i+1}$.

5. Let A be an $n \times n$ matrix.

   (a) State Gershgorin's theorem for localizing the eigenvalues of A.

   (b) Use Gershgorin's theorem to prove that a strictly diagonally dominant matrix does not have 0 as an eigenvalue and is therefore nonsingular.

6. Consider the following two-step method:

$$y_{k+1} + \alpha_0 y_k = h\left(\beta_1 f(t_k, y_k) + \beta_0 f(t_{k-1}, y_{k-1})\right)$$

for solving the initial-value problem $y'(t) = f(t, y)$.

   (a) Find $\alpha_0, \beta_0, \beta_1$ such that the method is **second** order.

   (b) Is the method **consistent**? If so why and if not why not?

   (c) Is the method **stable**? If so why and if not why not?

7. Show that if $||AB - I|| = \epsilon < 1$, then

$$||A^{-1} - B|| \le ||B|| \frac{\epsilon}{1-\epsilon} .$$

8. Consider the implicit numerical method

$$y_{i+1} = y_i + hf\left(\frac{y_i + y_{i+1}}{2}\right), \qquad i = 0, 1, 2, \ldots, N-1$$

for solving $\dfrac{dy(t)}{dt} = f(y(t))$, $y(0) = y_0$, for $0 \le t \le 1$, where $y_i \approx y(t_i), t_i = ih$, and $h = \dfrac{1}{N}$. Suppose that $|f(z_1) - f(z_2)| \le \dfrac{1}{3}|z_1 - z_2|$ for $z_1, z_2 \in \mathbb{R}$ and let $\max\limits_{0 \le t \le 1} |y'(t)| = M < \infty$. Prove that

$$|y_N - y(1)| \le chM$$

for a constant $c > 0$.

Hint: Note that $\dfrac{1+x}{1-x} \le 1 + 3x$ for $x \le \dfrac{1}{3}$.

9. Let $\displaystyle\int_0^1 f(x)\,dx \approx \sum_{i=0}^{N-1} f(x_i)h$ where $x_i = ih$ and $h = \dfrac{1}{N}$. Suppose that $f \in C^1[0,1]$. Prove that

$$\left|\int_0^1 f(x)\,dx - \sum_{i=0}^{N-1} f(x_i)\,h\right| \le \frac{h}{2} \max_{0 \le x \le 1} |f'(x)|$$

Hint: Note that $\displaystyle\int_0^1 f(x)\,dx = \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} f(x)\,dx$.

10. Let $f \in C^\infty(-\infty, \infty)$ and let $x_0 \in \mathbb{R}$ be given.

   (a) Prove that $C_h = \dfrac{f(x_0 + h) - f(x_0 - h)}{2h} = f'(x_0) + \displaystyle\sum_{i=1}^{\infty} c_i h^{2i}$ where $c_i, i = 1, 2, 3, \ldots$ are independent of h.

   (b) Suppose that C_h and $C_{\frac{h}{2}}$ have been calculated. Find constants α_1 and α_2 so that

$$\alpha_1 C_h + \alpha_2 C_{\frac{h}{2}} = f'(x_0) + O(h^4) .$$