Numerical Analysis Preliminary Examination, May 2008

Department of Mathematics and Statistics

Do nine of the following ten problems. Clearly indicate which nine are to be graded. Calculators are not allowed.

1. Let $p \geq 2$ and consider the continued fraction:

$$x = \cfrac{1}{p + \cfrac{1}{p + \cfrac{1}{p + \dots}}}.$$

This can be interpreted as $x = \lim_{n\to\infty} x_n$, where $x_1 = 1/p$, $x_2 = 1/(p + 1/p)$, and so forth, i.e., $x_{n+1} = 1/(p + x_n)$. Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ converges by using the Contraction Mapping Theorem. Also, find the value of x in terms of p.

2. Assume that the formula

$$\int_{-1}^{1} (1 + x^2)p(x)\, dx = \sum_{i=0}^{2} A_i p(x_i)$$

is exact for all polynomials $p(x)$ of degree less than or equal to 5. Find x_0, x_1, x_2 or find a polynomial $q(x) = x^3 + Bx^2 + Cx + D$ such that $q(x_i) = 0$ for $i = 0, 1, 2$.

3. Assume that $n \times n$ matrix A has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ with associated linearly independent eigenvectors $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$. Assume that $\lambda_1 = \lambda_2$ and that $|\lambda_1| = |\lambda_2| > |\lambda_3| \geq \dots \geq |\lambda_n|$. Describe the power method and prove that the power method converges to an eigenvector of λ_1 (or of λ_2).

4. Let A be an $n \times n$ nonsingular matrix. Recall that the LU factorization requires $\frac{2}{3}n^3 + O(n^2)$ arithmetic operations and one back-solve or one forward-solve requires $n^2 + O(n)$ operations.

(a) Describe an efficient algorithm for computing A^{-1} by solving n systems of equations. Show that the algorithm requires $\frac{8}{3}n^3 + O(n^2)$ arithmetic operations.

(b) Suppose that you wish to solve m linear systems $A\mathbf{x_j} = \mathbf{b_j}$, $j = 1, \dots, m$. By counting operations, explain whether it is more efficient to compute A^{-1} and then calculate $\mathbf{x_j} = A^{-1}\mathbf{b_j}$, $j = 1, \dots, m$ or to compute $A = LU$ and then solve $LU\mathbf{x_j} = \mathbf{b_j}$, $j = 1, \dots, m$.

5. Consider the initial-value problem $y'(t) = 3t^2, y(0) = 0$. Consider Euler's method $y_{n+1} = y_n + f(t_n, y_n)$ for $n = 0, 1, \dots N - 1$, with $y_0 = 0$ where $h = 1/n$ and $t_n = nh$. By calculating $y(1)$ and y_N, prove directly for this problem that the error satisfies

$$y(1) - y_n = \sum_{i=1}^{\infty} c_i h^i$$

where the coefficients c_i are independent of h. (Recall that $\sum_{k=1}^{M} k = M(M + 1)/2$ and $\sum_{k=1}^{M} k^2 = M(M + 1)(2M + 1)/6$.)

6. Let $e(h) = \int_0^1 f(x)\,dx - h\sum_{i=1}^{N} f(ih - h/2)$ where $h = 1/N$.
(a) If $f \in C^1[0,1]$, prove that there is a constant $c_1 > 0$ such that $|e(h)| \le c_1 h^1$.
(b) If $f \in C^2[0,1]$, prove that there is a constant $c_2 > 0$ such that $|e(h)| \le c_2 h^2$.

7. Let the 3×3 matrix A have eigenvalues $\lambda_1 = 2, \lambda_2 = 4, \lambda_3 = 6$, with associated eigenvectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$, respectively. Consider the iteration $\mathbf{x_{k+1}} = -(A - 3I)^{-1}(A - 5I)^{-1}\mathbf{x_k}$ where $\mathbf{x_0} = \mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}$. Prove that $\|\mathbf{x_k} - \mathbf{z}\| \le \frac{c}{3^k}$ where $\mathbf{z}$ is one of the vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ and c is a positive constant.

8. Consider the linear system $A\mathbf{x} = \mathbf{b}$ where $A = L + D + U$, L is strictly lower triangular, D is diagonal and U is strictly upper triangular. Assume that $L + 2D$ is nonsingular. Consider the iterative method

$$\mathbf{x_k} = (L + 2D)^{-1}(-2U - L)\mathbf{x_{k-1}} + 2(L + 2D)^{-1}\mathbf{b}.$$

(a) Assuming that $\mathbf{x_k} \to \mathbf{z}$ as $k \to \infty$, prove that $A\mathbf{z} = \mathbf{b}$.
(b) Let A be the $n \times n$ tridiagonal matrix

$$A = \begin{pmatrix} 2.1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 2.1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 2.1 & 1 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots & 1 & 2.1 & 1 \\ 0 & 0 & \dots & 0 & 1 & 2.1 \end{pmatrix}$$

where $a_{i,i} = 2.1$, $a_{i,j} = 1$ if $j = i - 1$ or $j = i + 1$, and $a_{i,j} = 0$ otherwise. Prove that the iterative method converges for this matrix.

9. Let $p(x)$ be the continuous piecewise linear interpolant to $f(x) = x^2$ on the interval $[0, 10]$ such that $p(k) = f(k)$ for $k = 0, 1, 2, \dots 10$. Let $e(x) = f(x) - p(x)$.
(a) Find $e(x)$ for $k - 1 \le x \le k$, for $k = 1, 2, \dots, 10$.
(b) Find $\max_{0 \le x \le 10} |e(x)|$.

10. Consider the initial-value problem $y'(t) = (t + 1)y(t)\cos^2(y(t)) + y(t)$ for $0 \le t \le 1$ with $y(0) = x$. Thus, $y(t) = y(t; x)$ is the solution when the initial condition is x. In particular, $y(1; x)$ is the value of $y(t)$ at $t = 1$ when the initial value is $y(0) = x$. Define $f(x) = 10 - y(1; x)$. Thus, the value of x that satisfies $f(x) = 0$ is the unique initial value so that $y(1) = 10$. Notice that $f(x) = 0$ is a nonlinear equation in x.
(a) Write down Newton's iteration and explain why Newton's method cannot be used to solve $f(x) = 0$ for this problem.
(b) Given that $x_1 = 1.5$, $f(x_1) = 2.73$, $x_2 = 1.7$, and $f(x_2) = 1.82$, calculate x_3 using the secant method.