Numerical Analysis Preliminary Examination, Aug 2009

Department of Mathematics and Statistics

Do **9** of the following **10** problems. Clearly indicate which **9** are to be graded. Calculators are not allowed.

1. Consider the function $q : \mathbb{R}^2 \to \mathbb{R}$. Assume that there exists a $z \in [a, b]$ such that $g(z, z) = z$ and g satisfies $|g(x, y) - g(\hat{x}, \hat{y})| \leq \lambda \max\{|x - \hat{x}|, |y - \hat{y}|\}$ for all $x, y, \hat{x}, \hat{y} \in [a, b]$, where $0 < \lambda < 1$. Consider the iterative method $x_{j+1} = g(x_j, x_{j-1})$ for $j = 2, 3, \cdots$, where $x_0, x_1 \in [a, b]$. Show that $|x_j - z| \to 0$ as $j \to \infty$ and z is the only point in $[a, b]$ such that $g(z, z) = z$.

2. Prove that if A is invertible and $\|B - A\| < \|A^{-1}\|^{-1}$, then B is invertible and

$$B^{-1} = A^{-1} \sum_{k=0}^{\infty} (I - BA^{-1})^k .$$

3. Prove that, if the matrix A is strictly diagonally dominant and Q is the lower triangular part of A, including the diagonal, then the Gauss-Seidel method

$$Qx^{(k)} = (Q - A)x^{(k-1)} + b, \quad k \geq 1,$$

converges to the solution of $Ax = b$, for any starting vector $x^{(0)}$.

4. Prove that if A is symmetric and positive definite, then the problem of solving $Ax = b$ is equivalent to the problem of minimizing the quadratic form

$$q(x) = x^T Ax - 2x^T b.$$

5. a) Show that the trace of a matrix A equals the sum of its eigenvalues. (Schur's Theorem may be useful).
b) Prove that if the eigenvalues of A satisfy $|\lambda_1| > |\lambda_i|$ for $i = 2, 3, \cdots, n$, then

$$\lambda_1 = \lim_{m \to \infty} \frac{tr(A^{m+1})}{tr(A^m)}$$

6. A natural cubic spline S on $[0, 2]$ is defined by

$$S_0(x) = 1 + 2x - x^3 \qquad \text{on } 0 \leq x < 1,$$
$$S_1(x) = 2 + b(x - 1) + c(x - 1)^2 + d(x - 1)^3 \qquad \text{on } 1 \leq x \leq 2.$$

Find b, c and d.

7. Let $e(h) = \int_0^1 f(x)dx - h\sum_{i=1}^N f(ih - \frac{h}{2})$, where $h = 1/N$
   a) For $f \in C^1[0,1]$, prove that for all N there exists a constant $c_1 > 0$ such that
$$|e(h)| \le c_1 h.$$
   b) For $f \in C^2[0,1]$, prove that for all N there exists a constant $c_2 > 0$ such that
$$|e(h)| \le c_2 h^2.$$

8. Suppose that x_i and A_i, for $i = 0, 1, 2$, are selected so that the quadrature formula
$$\int_{-1}^{1} x^2 f(x)dx \approx \sum_{i=0}^{2} A_i f(x_i),$$
   is exact for any polynomial of degree 5. Find the third degree polynomial $q_3(x)$ such that $q_3(x_i) = 0$, for $i = 0, 1, 2$.

9. Consider the initial value problem
$$\begin{cases} x'(t) = f(t,x) \\ x(t_0) = x_0 \end{cases}$$
   Show that, if w_1, w_2, α and β satisfy
$$\begin{cases} w_1 + w_2 = 1 \\ w_2\alpha = \frac{1}{2} \\ w_2\beta = \frac{1}{2} \end{cases}$$
   then
$$\begin{cases} x(t+h) = x(t) + (w_1F_1 + w_2F_2) + O(h^3) \\ F_1 = hf(t,x); \\ F_2 = hf(t + \alpha h, x + \beta F_1) \end{cases}$$

10. Show how the Shooting method can be used to solve the two-point boundary value problem of the following type, in which the constants α, β and c_{ij} and the functions $u(t)$, $v(t)$ and $w(t)$ are all given:
$$\begin{cases} x'' = u(t) + v(t)x + w(t)x' \\ c_{11}x(a) + c_{12}x'(a) = \alpha \\ c_{21}x(b) + c_{22}x'(b) = \beta \end{cases}$$
   Assume that the solution exists and is unique.