Numerical Analysis Preliminary Examination, May 2009

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Do eight of the following nine problems. Clearly indicate which eight are to be graded. Calculators are not allowed.

1. Let $F(x)$ have a fixed point s, and assume that there exists an integer $q \geq 2$ such that $F^{(k)}(s) = 0$ for $1 \leq k \leq q-1$, but $F^{(q)}(s) \neq 0$. Prove that the sequence $[x_n]$, defined by $x_{n+1} = F(x_n)$ converges to the fixed point s with order of convergence q. Assume $F(x)$ to be smooth enough.

2. Consider the iterative procedure:
$$\vec{y}_{j+1}^{(k+1)} = \vec{y}_j + \frac{h}{24}\left[9\vec{f}(\vec{y}_{j+1}^{(k)}) + 19\vec{f}(\vec{y}_j) - 5\vec{f}(\vec{y}_{j-1})\right],$$
where $\vec{y}_{j+1}^{(k+1)}$, $\vec{y}_{j+1}^{(k)}$, $\vec{y}_j$, and $\vec{y}_{j-1} \in \mathbb{R}^n$, $\vec{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\vec{y}_j$ and $\vec{y}_{j-1}$ are given. Assume that $\|\vec{f}(\vec{z}) - \vec{f}(\vec{w})\|_\infty \leq \frac{8}{9}\|\vec{z} - \vec{w}\|_\infty$ for all $\vec{z}$ and $\vec{w} \in \mathbb{R}^n$. Prove that if h is sufficient small, then $\|\vec{f}(\vec{y}_{j+1}^{(k+1)}) - \vec{f}(\vec{y}_{j+1})\|_\infty \longrightarrow 0$ as $k \longrightarrow \infty$, where $\vec{y}_{j+1}$ satisfies
$$\vec{y}_{j+1} = \vec{y}_j + \frac{h}{24}\left[9\vec{f}(\vec{y}_{j+1}) + 19\vec{f}(\vec{y}_j) - 5\vec{f}(\vec{y}_{j-1})\right].$$

3. Let $\|\cdot\|$ be any induced matrix norm. Prove that if E is an $n \times n$ matrix for which $\|E\|$ is sufficiently small, then
$$\|(I-E)^{-1} - (I+E)\| \leq 3\|E\|^2.$$
Determine how small $\|E\|$ should be.

4. Let the $n \times n$ matrix A be diagonalized by similarity transformation $D = P^{-1}AP$. Consider any $n \times n$ matrix B and let $C = P^{-1}BP$.
a) Show that $A + B$ and $D + C$ have the same eigenvalues.
b) Show that the eigenvalues of $A + B$ lie in the union of the disks
$$\{\lambda \in \mathbb{C} : |\lambda - \lambda_i| \leq \kappa_\infty(P)\|B\|_\infty\} \quad (1 \leq i \leq n)$$
where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, and $\kappa_\infty(P)$ is the condition number of P.

5. Let the 3×3 matrix A have eigenvalues $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = 8$ with corresponding eigenvectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$. Consider the iterative method
$$\vec{x}_{k+1} = \frac{1}{3}(I - A)(A - 5I)^{-1}\vec{x}_k,$$
where $\vec{x}_0 = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$. Prove that $\|\vec{x}_k - \vec{z}\| \longrightarrow 0$ as $k \longrightarrow \infty$, where $\vec{z}$ is one of the eigenvectors $\vec{v}_1$, $\vec{v}_2$ or $\vec{v}_3$.

6. Suppose that $f(x)$ satisfies a Lipschitz condition $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in [0, 1]$. Let $\Psi(x)$ be a piecewise constant approximation to $f(x)$ such that

$$\Psi(x) = \frac{f(x_i) + f(x_{i+1})}{2}, \text{ for } x_i \leq x < x_{i+1}, \text{ for } i = 0, 1, \ldots, N-1$$

with $x_i = ih$ and $h = 1/N$. Prove that

$$\max_{0 \leq x \leq 1} |\Psi(x) - f(x)| \leq c\, h$$

for some constant c.

7. Suppose we wish to approximate an odd function by a polynomial of degree $\leq n$ (n odd) using the norm $\|f\| = [\int_{-a}^{a} |f(x)|^2 w(x) dx]^{1/2}$, where $w(x)$ is a even positive weight function. Prove that the best approximation is also odd.

8. a) Find the constants A and B such that the formula

$$\int_0^{2\pi} f(x) dx \approx A\, f(0) + B\, f(\pi),$$

is exact for any function of the form $f(x) = a + b\cos x$.
b) Prove that the resulting formula is exact for any function of the form

$$f(x) = \sum_{k=0}^{n} [a_k \cos((2k+1)x) + b_k \sin(kx)].$$

9. Consider the initial-value problem $dy/dt = 1 + 2t + 3\cos(y(t))$, $0 \leq t \leq 1$, with $y(0) = 1$. Suppose that the solution satisfies $\max_{0 \leq t \leq 1} |y''(t)| = M < \infty$. Consider the approximation

$$y_{k+1} = y_k + h(1 + 2t_k + 3\cos(y_k))$$

for $k = 0, 1, 2, \cdots, N-1$, $y_0 = y(0) = 1$, and $h = 1/N$ and $t_k = kh$. Prove that $\|y(1) - y_N\| \leq \frac{Mh}{6}(e^3 - 1)$.