Numerical Analysis Preliminary Examination, August 2010

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Do **five** of the six problems. Clearly indicate which five are to be graded. Calculators are forbidden.

1. Consider *Heun's method* for the approximate solution of a scalar IVP $y' = f(x, y)$, $f \in C^{(2,2)}$, with initial conditions $y(x_0) = y_0$ and stepsize h. A step of Heun's method is given by

$$\tilde{y}_n = y_n + hf(x_n, y_n)$$
$$x_{n+1} = x_n + h$$
$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, \tilde{y}_n)\right].$$

   (a) State carefully definitions of the following:
      i. Absolute stability
      ii. A-stability
      iii. Local truncation error

   (b) Find the real part of the region of absolute stability for Heun's method. Can you determine from this whether Heun's method A-stable?

   (c) Prove that the local truncation error of Heun's method is $O(h^3)$

2. On the space of real-valued functions on $[-1, 1]$ let there be defined the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$$

and associated norm $\|\cdot\|_2 = \sqrt{\langle\cdot,\cdot\rangle}$. Let L^2 be the space of functions for which $\|\cdot\|_2$ is finite, and let P^N be the space of polynomials of degree $\le N$. For any $f \in L^2$, let $p_{N,f}(x) \in P^N$ be the polynomial of degree $\le N$ that best approximates f in the $\|\cdot\|_2$ norm (*i.e.*, $p_{N,f}$ is the least-squares N-th degree polynomial approximation to f).

   (a) State the Weierstrass approximation theorem.

   (b) Let $f(x) = x^3$. Compute $p_{1,f}(x)$.

   (c) Prove that for any $f \in C^0[-1, 1]$ the sequence of least-squares polynomial approximations $p_{N,f}$ converges in the $\|\cdot\|_2$ norm, i.e.,

$$\lim_{N\to\infty} \|p_{N,f} - f\|_2 = 0.$$

   (d) Consider the proposition: *for any fixed degree N and for any* $f \in C^1[-1, 1]$, $p_{N,f'}(x) = p'_{N,f}(x)$, where the prime indicates differentiation. Prove or give a counterexample.

3. Consider finding a root x^* of $f(x) = 0$, where $f : \mathbb{R}^n \to \mathbb{R}^n$.

(a) State what it means for a sequence $\{x_k\}_{k=0}^{\infty}$ to converge with order p in a normed space.

(b) Suppose Newton's method is used to find the root of the univariate function $f(x) = x^2$ starting from an initial guess x_0.

i. Show that for any choice of x_0, the convergence to $x^* = 0$ is linear, *i.e.*, with order $p = 1$.

ii. When certain conditions are met, Newton's method will converge quadratically (*i.e.* with $p = 2$). Show where in this problem these conditions are violated, thus explaining the loss of quadratic convergence.

(c) Let A be a nonsingular $m \times m$ matrix and $\|\cdot\|$ be a matrix norm. The problem of computing A^{-1} can be posed as finding the nonzero root of the equation $f(X) = XAX - X = 0$. (The lack of a transpose in the first term is correct, not a typographical error). You are given that Newton's method together with a certain cleverly-chosen approximation to the Jacobian leads to the sequence

$$X_{k+1} = X_k \left[2I - AX_k\right].$$

Prove that this sequence converges quadratically to A^{-1} provided $\|AX_0 - I\| \leq 1$.

4. Suppose you wish to approximate an integral by a two-point quadrature rule,

$$\int_{-1}^{1} f(x)\,dx \approx Q_2(f) = w_1 f(x_1) + w_2 f(x_2).$$

(a) Supposing that you have (for some reason) chosen to use the nodes $x_1 = -\frac{1}{2}$, $x_2 = 1$, find weights such that Q_2 is exact for all linear functions.

(b) Find nodes and weights for a two-point rule Q_2 that is exact for all cubic polynomials.

(c) Use the rule derived in part (b) to approximate

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{x^2}\,dx.$$

Note the change in limits of integration.

5. Suppose a matrix $A \in \mathbb{C}^{n\times n}$ has distinct eigenvalues $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$ and associated linearly independent eigenvectors $v_1, v_2, \cdots, v_n$.

   (a) Carefully describe the power method for approximation of λ_1 and v_1.

   (b) Prove that in exact arithmetic and given certain conditions on the initial guess $v_1^{(0)}$ (these conditions to be specified as part of your proof) the power method converges to the eigenpair λ_1, v_1.

   (c) In practice, the power method will nearly always converge to λ_1, v_1 for any $v_1^{(0)} \neq 0$, even if $v_1^{(0)}$ does not meet the conditions stated in part (b). Why?

   (d) Were the eigenvalue λ_n desired instead of λ_1, how would you modify the power method to find λ_n?

6. (a) For each of the matrix factorizations LU, Cholesky, QR, SVD:

      i. State clearly the class of matrices for which the factorization exists.

      ii. Describe the properties of the factors.

   (b) Let $A \in \mathbb{C}^{m\times n}$ be full rank and have the full QR factorization

$$A = \begin{bmatrix} Q & Q_\perp \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix},$$

   where $Q_\perp$ has orthonormal columns and is orthogonal to Q. Suppose R has the singular value decomposition

$$R = W\Sigma V^*,$$

   where V^* indicates the conjugate transpose of V. Prove that

$$\begin{bmatrix} QW & Q_\perp \end{bmatrix} \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^*$$

   is a singular value decomposition of A.