Numerical Analysis Preliminary Examination, Spring 2010

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Do **five** of the following **six** problems. Clearly indicate which five are to be graded. Calculators are not allowed.

1. Let the inner product $\langle f, g \rangle$ be defined by

$$\langle f, g \rangle = \int_0^\infty e^{-x} f(x) g(x)\, dx.$$

With this inner product one can generate an orthonormal family of polynomials called the Laguerre polynomials, the first three of which are

$$L_0(x) = 1, \quad L_1(x) = 1 - x, \quad L_2(x) = \frac{1}{2}\left(x^2 - 4x + 2\right).$$

Let P^2 be the space of quadratic polynomials, and let the norm $\|\cdot\|_2$ be defined by $\|f\|_2 = \sqrt{\langle f, f \rangle}$.

(a) Let $g(x) = e^{-x/2}$. Find the polynomial $u \in P^2$ that best approximates g in the $\|\cdot\|_2$ norm. You may leave any definite integrals unevaluated.

(b) Prove that there is no uniformly-convergent sequence of polynomial approximations to $g(x) = e^{-x/2}$ on the interval $[0, \infty)$.

(c) Explain why the conclusion in part (b) does not violate the Weierstrass approximation theorem.

2. Consider the scalar initial value problem $y' = \phi(t, y)$, $y(0) = y_0$. The backward Euler (BE) method steps from (t_n, y_n) to (t_{n+1}, y_{n+1}) using the rule

$$y_{n+1} = y_n + h\phi(t_{n+1}, y_{n+1})$$

where $h = t_{n+1} - t_n$.

(a) Give the definitions of the following terms:

  i. absolute stability

  ii. set of absolute stability

  iii. A-stability

(b) Find the set of absolute stability for the BE method. Is the BE method A-stable?

(c) Given certain conditions on the differentiability of ϕ, the truncation error in a single BE step is $O(h^2)$. Clearly state the required differentiability conditions and prove the result.

3. Suppose $Q(f)$ is any numerical integration rule for the interval $[0,1]$,

$$Q(f) = \sum_{j=0}^{M} w_j f(x_j)$$

that integrates constants exactly. Suppose that a composite quadrature rule for $[a,b]$ has been formed by replicating this basic rule on N subintervals of size $h = \frac{b-a}{N}$,

$$Q_N(f) = h \sum_{k=0}^{N-1} \sum_{j=0}^{M} w_j f(x_{k,j})$$

where $x_{k,j} = a + kh + x_j$.

(a) Assume that f is Lipschitz continuous on $[a,b]$ and prove that

$$E_N(f) = \left| \int_a^b f(x)\,dx - Q_N(f) \right| \to 0 \quad \text{as } N \to \infty.$$

(b) Prove the same result as in (a) assuming only that $f \in C^0[a,b]$.

4. In this problem, a tilde ($\sim$) indicates a computed quantity. For example, if R is the upper triangular matrix that would be obtained from a QR factorization in exact arithmetic, $\tilde{R}$ is the matrix actually computed by factorization on a machine with roundoff error. The machine epsilon is denoted ϵ_m and $\|\cdot\|$ is any induced norm.

(a) Let M be the 3×2 matrix

$$M = \begin{pmatrix} -2 & 2 \\ -2 & 1 \\ -1 & 0 \end{pmatrix}.$$

Compute a QR factorization of M. Use the algorithm of your choice. Clearly state which algorithm you are using and whether your result is a full or reduced factorization.

(b) Let A be an $n \times n$ matrix. Consider the following three-step algorithm for solving $Ax = b$, with stability results given for each step of the algorithm.

i. A is factored into QR via a method that preserves unitarity of Q, *i.e.*, $\tilde{Q}$ is exactly unitary in machine arithmetic. You are given that this operation is backward stable: there exists a matrix δA such that $\tilde{Q}\tilde{R} = A + \delta A$, with $\frac{\|\delta A\|}{\|A\|} = O(\epsilon_m)$.

ii. The operation $y = Q^* b$ is performed, where Q^* is the conjugate transpose of Q. You are given that this operation is backward stable: there exists a matrix δQ such that $\left(\tilde{Q} + \delta Q\right)\tilde{y} = b$, with $\|\delta Q\| = O(\epsilon_m)$.

iii. The triangular system $Rx = y$ is solved by backsubstitution. You are given that this operation is backward stable: there exists a matrix δR such that $\left(\tilde{R} + \delta R\right)\tilde{x} = \tilde{y}$, with $\frac{\|\delta R\|}{\|\tilde{R}\|} = O(\epsilon_m)$.

Prove that this algorithm is backward stable, *i.e.*, that there exists a matrix ΔA such that

$$(A + \Delta A)\,\tilde{x} = b$$

and

$$\frac{\|\Delta A\|}{\|A\|} = O(\epsilon_m).$$

5. In this problem all matrices are real.

(a) Let A be a full rank $m \times n$ matrix with the reduced (or thin) singular value decomposition (SVD) $A = U\Sigma V^T$ given. Consider X and Y,

$$X = \left(A^T A\right)^{-1}$$

$$Y = A\left(A^T A\right)^{-1} A^T.$$

Find the SVDs of X and Y in terms of U, V, and Σ.

(b) Let A be $m \times n$ with reduced SVD $A = U\Sigma V^T$. Show that the singular values σ of A can be obtained by solving the eigenvalue problem $A^T A x = \sigma^2 x$, and that the eigenvectors x are the columns of V.

(c) Given A as in part (b), find an eigenvalue problem whose eigenvectors are the columns of U.

(d) Let A be $n \times n$ symmetric and B be $n \times n$ symmetric positive definite. Use an appropriate factorization to find the symmetric matrix C such that the generalized eigenvalue problem $Ax = \lambda Bx$ is transformed to the symmetric eigenvalue problem $Cy = \lambda y$.

(e) Let A and B be defined as in part (d). A naive approach to the generalized eigenvalue problem $Ax = \lambda Bx$ is to transform it to the equivalent system

$$B^{-1} A x = \lambda x.$$

However, this form loses the symmetry of the original problem.
For purposes of numerical calculation, why is it important to transform to a symmetric problem?

6. Consider the problem of finding the real roots of the function $f(x) = x - \frac{1}{2}\cos x$.

(a) Let M be a real number greater than $\frac{1}{2}$. Prove that the interval $[-M, M]$ contains exactly one root and that the fixed-point iteration $x_{n+1} = \frac{1}{2}\cos x_n$ converges to that root given any initial point $x_0 \in [-M, M]$.

(b) Estimate the root by carrying out a single step of Newton's method starting at $x_0 = \frac{\pi}{2}$.