Numerical Analysis Prelim, August 2011

Work 6 of the following 7 problems. If you attempt all 7, mark clearly which problems you wish to be graded.

1. Given an arbitrary $n \times n$ matrix A, prove that if z is an eigenvalue of $A + \delta A$ for some δA with $\|\delta A\|_2 \le \epsilon$, then there exists a vector u with $\|(A - zI)u\|_2 \le \epsilon$ and $\|u\|_2 = 1$.

2. (a) For each of the matrix factorizations LU, Cholesky, QR, SVD:
   i. State clearly the class of matrices for which the factorization exists.
   ii. Describe the properties of the factors.

   (b) Let $A \in \mathbb{C}^{m \times m}$ be a nonsingular matrix. Recall that the LU factorization requires asymptotically $\frac{2}{3}m^3$ flops, and that one forward-solve or one back-solve requires asymptotically m^2 flops.
   i. Describe an efficient algorithm for computing A^{-1} by solving m systems of equations. Show that this algorithm requires asymptotically $\frac{8}{3}m^3$ flops.
   ii. Suppose that you wish to solve k linear systems $Ax_j = b_j$ for $j = 1, \ldots, k$. By counting operations, explain whether it is more efficient to compute A^{-1} and then calculate $x_j = A^{-1}b_j$, $j = 1, \ldots, k$ or to compute $A = LU$ and then solve $LUx_j = b_j$, $j = 1, \ldots, k$.

3. Let $f(x)$ be continuous on $[0, 1]$, and let P^N be the space of N-th degree polynomials. Let there be defined the L^2 norm

$$\|v\|_2 = \sqrt{\int_0^1 v(x)^2 \, dx}$$

   and let $p_n(x)$ be the n-th degree polynomial that best approximates f in the L^2 norm.

   (a) Prove that

$$\|f - p_{n+1}\|_2 \le \|f - p_n\|_2 \,.$$

   (b) Prove the sequence $\{p_n\}_{n=0}^{\infty}$ converges to f in the L^2 norm.

   (c) Prove or disprove the proposition: there must exist at least one point $a \in (0, 1)$ such that $f(a) = p_1(a)$.

4. Consider the problem of finding the real roots of the function $f(x) = x - \frac{1}{2}\cos x$.

   (a) Let M be a real number greater than $\frac{1}{2}$. Prove that the interval $[-M, M]$ contains exactly one root and that the fixed-point iteration $x_{n+1} = \frac{1}{2}\cos x_n$ converges to that root given any initial point $x_0 \in [-M, M]$.

   (b) Estimate the root by carrying out a single step of Newton's method starting at $x_0 = \frac{\pi}{2}$.

5. Let

$$Q_N(f) = \sum_{i=1}^{N} w_{N,i} f(a_{N,i})$$

   be a quadrature rule for approximating integrals of the form

$$I(f) = \int_{-1}^{1} f(x) \, dx.$$

(a) Prove the following theorem: If Q_N is exact for all polynomials of degree N, and if all weights $w_{N,i}$ are positive, then for all $f \in C^0[-1,1]$ it is the case that

$$Q_N(f) \to I(f) \quad \text{as } N \to \infty.$$

(b) For each of the following types of quadrature rules, are the hypotheses of exactness and of positive weights appearing in the theorem of part (a) satisfied? If not, state which hypotheses do not hold. A brief explanation is acceptable in place of a proof.

i. N-point Gaussian quadrature (with any positive weight function)

ii. N-point Newton-Cotes (open or closed)

iii. The N-point composite trapezoidal rule

6. Consider the scalar IVP $y' = f(t,y), y(0) = y_0$. A trapezoidal rule (TR) step for this problem is

$$t_{n+1} = t_n + h$$

$$y_{n+1} = y_n + \frac{1}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})]$$

where h is the stepsize.

(a) Give careful definitions of the following terms:

i. absolute stability

ii. A-stability

iii. order of accuracy

(b) Prove that the TR method is A-stable.

(c) Prove that the TR method has order of accuracy 2.

(d) One can construct an explicit method based on the TR step by replacing y_{n+1} in the right-hand side of the TR step formula by an approximation $\tilde{y} \approx y_{n+1}$. With such an approximation, the step becomes

$$y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, \tilde{y})].$$

To what accuracy must $\tilde{y}$ be computed to ensure that this explicit method has the same order of accuracy as the implicit TR method?

7. Consider the nonlinear boundary value problem

$$-u'' = \frac{1}{2}(1 + \sin u)$$

posed on (0,1) with boundary conditions $u(0) = u(1) = 0$.

(a) Write down the nonlinear algebraic system of equations resulting from the finite difference method with N internal nodes.

(b) Consider the iterative strategy of solving the nonlinear system with an initial solution vector $u^{(0)}$ and iterating $Au^{(n+1)} = F(u^{(n)})$, where A is the finite difference matrix obtained by discretizing $-u''$ and $\left(F(u^{(n)})\right)_i = \frac{1}{2}\left(1 + \sin u_i^{(n)}\right)$. Show that this iteration converges to the solution of the algebraic equation for any initial input. You may find the fact that the eigenvalues of A are known to be $\left\{2\left(1 - \cos\left(\frac{\pi j}{N+1}\right)\right)\right\}_{j=1}^{N}$ helpful in determining the norm of A^{-1} and/or A.