Numerical Analysis Prelim, May 2011

Work 6 of the following 7 problems. If you attempt all 7, mark clearly which problems you wish to be graded.

1. (a) State what it means for a sequence in a normed space to converge with order p.

   (b) Suppose that $f : D \to \mathbb{R}$ is some continuous function defined on an open interval $D \subset \mathbb{R}$. State conditions under which f has a fixed point in D. Under what conditions is the fixed point unique?

   (c) Suppose that $f \in C^m(D)$ has a fixed point x_0 such that $f'(x_0) = f''(x_0) = \cdots = f^{(m-1)}(x_0) = 0$. Show that successive approximations converge to the fixed point with order m.

   (d) For differentiable f, Newton's method for root finding can be thought of as finding a fixed point x of $g(x) = x - \frac{f(x)}{f'(x)}$. Find suitable conditions on f such that the result from (c) implies that convergence of Newton's method must be quadratic.

2. Consider the nonlinear boundary value problem

$$-u'' = \cos u$$

   posed on (0,1) with boundary conditions $u(0) = 0 = u(1)$.

   (a) Write down the nonlinear algebraic system of equations resulting from the finite difference method with N internal nodes.

   (b) Consider the iterative strategy of solving the nonlinear system from (a) with an initial solution vector $u^{(0)}$ and iterating $Au^{(n+1)} = F(u^{(n)})$, where A is the finite difference matrix obtained by discretizing $-u''$ and $\left(F(u^{(n)})\right)_i = \cos u_i^{(n)}$. Show that this iteration converges to the solution of the algebraic equation for any initial input. (Hint: the fact that the eigenvalues of A are known to be $\left\{2\left(1-\cos\left(\frac{\pi j}{N+1}\right)\right)\right\}_{j=1}^{N}$ may be helpful in determining the norm of A^{-1} and/or A.)

3. (a) Let A be an arbitrary $n \times n$ matrix and $\|\cdot\|$ be an arbitrary matrix norm. Prove that $\lim_{k\to\infty} \|A^k\| = 0$ if and only if $\rho(A) < 1$, where $\rho(A)$ is the spectral radius of A.

   (b) Let A be an $n \times n$ matrix with the properties that $A^*A = AA^*$ and that the n eigenvalues of A^*A are distinct. Prove that $\|A\|_2 = \rho(A)$, where $\rho(A)$ is the spectral radius of A.

4. (a) For each of the matrix factorizations LU, Cholesky, QR, SVD:

      i. State clearly the class of matrices for which the factorization exists.

      ii. Describe the properties of the factors.

   (b) If A is nonsingular and has an LU decomposition, taking the convention that L is unit lower triangular, prove that the decomposition is unique.

   (c) Let A be a nonsingular square matrix. Let $A = QR$ be the QR factorization of A, and let $A^*A = U^*U$ be a Cholesky factorization of the normal matrix A^*A. Does $R = U$? Prove it or give a counter example.

5. Let P^2 be the space of quadratic polynomials on $\mathbb{R}$, and let x_1, x_2, x_3 be three distinct nodes. The second-degree Lagrange basis functions $\ell_i(x)$ for this set of nodes are defined implicitly by

$$\ell_i(x) \in P^2$$
$$\ell_i(x_j) = \delta_{ij}$$

where δ_{ij} is the Kronecker delta.

(a) Given the nodes $0, \frac{1}{2}, 1$ find the functions $\ell_i(x)$.

(b) Prove that for any $f \in C^0[0,1]$, the polynomial

$$p(x) = \sum_{i=1}^{3} f(x_i)\,\ell_i(x)$$

is the unique element of P^2 such that $p(x_i) = f(x_i)$ for $i = 1, 2, 3$.

(c) Prove or disprove the proposition: for any $g \in C^1[0,1]$, there is a unique quadratic polynomial $q(x)$ such that

$$q(0) = g(0)$$
$$q'\left(\frac{1}{2}\right) = g'\left(\frac{1}{2}\right)$$
$$q(1) = g(1).$$

6. Let

$$Q_N(f) = \sum_{n=1}^{N} w_n f(a_n)$$

be a quadrature rule for approximating integrals of the form

$$I(f) = \int_0^1 x f(x)\,dx.$$

(a) Derive the one-point rule

$$Q_1(f) = w_1 f(a_1)$$

that is exact for all $f \in P^1$, where P^1 is the space of linear polynomials on $\mathbb{R}$.

(b) Derive Find the two-point rule

$$Q_2(f) = w_1 f(a_1) + w_2 f(a_2)$$

that is exact for all $f \in P^3$, where P^3 is the space of cubic polynomials on $\mathbb{R}$.

7. Consider the scalar initial value problem $y' = f(t, y)$, $y(0) = y_0$.

(a) Give careful definitions of the following terms:

i. absolute stability

ii. A-stability

iii. order of accuracy

(b) Either construct an A-stable explicit Runge-Kutta method of any order $p \geq 1$ you find convenient, or prove the impossibility of such a construction.