Ph.D. Preliminary Exam in Numerical Analysis
Aug 2012

Do all seven problems

1. Let A be a real symmetric matrix.

(**a**) Prove that A has a Cholesky factorization if and only if A is positive definite.

(**b**) Assuming A has a Cholesky factorization, find the number of operations (to leading order) required to compute that factorization.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x - \frac{3}{4}e^{-x}$. Prove that $f(x) = 0$ has a unique solution x^*, and that the fixed-point iteration $x_{n+1} = \frac{3}{4}e^{-x_n}$ converges to x^* from any initial $x_0 \in \mathbb{R}$.

3. Let f be any non-constant C^∞ function on $\mathbb{R}$, and let $f_h'(x_0)$ be the centered difference approximation to $f'(x_0)$ computed with stepsize h,

$$f_h'(x_0) = \frac{f(x_0 + h) - f(x_0 - h)}{2h}.$$

(**a**) Suppose $f_h'(x_0)$ is computed in exact arithmetic. Prove that $f_h'(x_0) \to f'(x_0)$ as $h \to 0$.

(**b**) Suppose $f_h'(x_0)$ is computed in idealized floating point arithmetic with specified ϵ_m.

i. Prove that $f_h'(x_0)$ does not converge as $h \to 0$.

ii. Find the step h^* that minimizes the absolute error $\left| f'(x_0) - f_h'(x_0) \right|$.

4. Consider the *midpoint method* for the approximate solution of a scalar initial value problem $y' = f(x, y)$, $f \in C^{(1,1)}$, with initial conditions $y(x_0) = y_0$ and stepsize h. A step of the midpoint method is given by

$$\tilde{y}_n = y_n + \frac{1}{2}hf(x_n, y_n)$$

$$\tilde{x}_n = x_n + \frac{h}{2}$$

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + hf(\tilde{x}_n, \tilde{y}_n).$$

(**a**) State carefully definitions of the following:

i. Absolute stability

ii. A-stability

iii. Local truncation error

(**b**) Find the region of absolute stability for the midpoint method. Is the midpoint method A-stable?

(**c**) Prove that the local truncation error of the midpoint method is $O(h^3)$.

5. Consider approximation of the integral $I(f) = \int_{-1}^{1} f(x)\, dx$.

   (a) Find a three-point quadrature rule Q that computes $I(f)$ exactly for all $f \in P^5$, where P^5 is the space of all polynomials of degree at most 5.

   (b) Derive a bound on the error $I(f) - Q(f)$ in terms of $\left\| f^{(k)} \right\|_{\infty}$. State as needed any assumptions about k, the degree of differentiability of f.

6. Let A be any $M \times N$ complex matrix. The notation A^H denotes the conjugate transpose of A.

   (a) Describe the properties (size, shape, structure, or other notable attributes) of the factors of the singular value decomposition (SVD) of A. When appropriate, distinguish between the full and reduced SVDs.

   (b) Use the SVD of A to compute the condition number (with respect to the Euclidean norm) of $A^H A$.

   (c) Prove that for every $\epsilon > 0$, the matrix $\epsilon I + A^H A$ is Hermitian positive definite.

7. Let the inner product $(\cdot, \cdot)$ be defined by

$$(u, v) = \int_{-1}^{1} u(x)\, v(x)\, dx,$$

and let $\|\cdot\|$ be the norm induced by that inner product: $\|v\| = \sqrt{(v, v)}$. Let P^N be the space of polynomials of degree at most N.

   (a) Find the first-degree polynomial $u_1 \in P^1$ that best approximates in the $\|\cdot\|$ norm the function $f(x) = x^5$.

   (b) Let f be continuous on $[-1, 1]$, and let u_N be the N-th degree polynomial that best approximates f in the $\|\cdot\|$ norm. Prove that $u_N \to f$ uniformly as $N \to \infty$.