Numerical Analysis Prelim, May 2012

1. Suppose you are asked to solve M systems $Ax_k = b_k$, $k = 1$ to M. The matrix A is N by N and nonsingular. Estimate the cost of each of the following approaches to the calculation:

   (a) Factor $A = LU$ once, then use triangular solves against the factors to solve the M systems.

   (b) Factor $A = LU$ once, then use triangular solves against the factors to find A^{-1}, then use multiplication by A^{-1} to solve the M systems

   (c) For each of the M systems, factor $A = LU$ from scratch, then use the factors to solve that system.

   Use only the leading-order term in the cost estimates.

2. Let A be a nonsingular $n \times n$ matrix, b a vector in $\mathbb{R}^n$, and $\alpha \neq 0$ a real constant. Consider the modified Richardson iteration

$$x^{(k+1)} = x^{(k)} + \alpha \left(b - Ax^{(k)} \right).$$

   (a) Find the fixed point x^* of this iteration, and show that it is unique.

   (b) Show that for any nonsingular symmetric A there is an $\alpha > 0$ such that this iteration converges.

3. Let $\{x_i\}_{i=1}^{N}$ be a set of uniformly-spaced points in an interval $[a, b]$. You may assume that they are listed in increasing order, and that $x_1 = a$, $x_N = b$.

   (a) Describe a basis for N-point piecewise linear Lagrange interpolation (PWLL) on $[a, b]$

   (b) Let $u_N(x)$ be the PWLL approximation to $f \in C^2[a, b]$ using the N-point uniform grid. Prove that $\|u_N - f\|_\infty \to 0$.

4. Consider integrals of the form

$$I(v) = \int_{-1}^{1} \frac{v(x)\, dx}{\sqrt{1 - x^2}}.$$

   (a) Find nodes and weights for a Gaussian quadrature rule Q that computes I exactly for all $v \in P^3$, where P^3 is the space of cubic polynomials.

   (b) Derive a bound on the error $|I(f) - Q(f)|$ in terms of $\left\| f^{(k)} \right\|_\infty$ for some k to be specified by you as needed.

5. Consider the trapezoidal rule (TR) for the approximate solution of a scalar IVP $y' = f(x, y)$, $f \in C^{(2,2)}$, with initial conditions $y(x_0) = y_0$ and stepsize h. A step of the TR is given by

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + \frac{h}{2} \left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}) \right].$$

   (a) State carefully definitions of the following:

      i. Absolute stability

      ii. A-stability

      iii. Local truncation error

   (b) Find the region of absolute stability for the TR. Is the TR A-stable?

   (c) Prove that the local truncation error of the TR is $O(h^3)$.

6. Given any $r > 0$, Heron's algorithm for computing $\sqrt{r}$ is

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{r}{x_n}\right)$$

starting from a suitable initial guess x_0.

(a) Show that $\sqrt{r}$ is a fixed point of this iteration.
(b) Derive Heron's algorithm as a special case of Newton's method.
(c) Prove convergence of Heron's algorithm for all $x_0 \in [a, b]$, where $[a, b]$ is an interval (with $|b - a| > 0$) to be chosen by you as needed.

7. Let $\|\cdot\|_2$ be the Euclidean vector norm on $\mathbb{C}^N$.

(a) State the definition of the matrix norm subordinate to $\|\cdot\|_2$.
(b) Let A be an $N \times N$ complex matrix. Prove that $\|A\|_2$ is the largest singular value of A.
(c) Suppose a matrix A is singular. Prove that $A^H A + \alpha I$ is nonsingular for any $\alpha > 0$.