Numerical Analysis Preliminary Examination, August 2013

Work **all six** of the following problems.

1. Consider a nonsingular matrix $A \in \mathbb{C}^{n\times n}$.

   (a) Write out the steps of the ("pure" unshifted) QR algorithm for finding the eigenvalues of A.

   (b) Show that each of the matrices A_k generated by the QR algorithm is unitarily similar to A.

   (c) Show that if A is upper Hessenberg, then so are each of the matrices A_k generated by the QR algorithm.

   (d) Describe an algorithm for computing an SVD of A using the eigenvalue decomposition of a related matrix.

2. Let A be the 3×3 matrix be

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 5 \\ 3 & 5 & 14 \end{bmatrix}$$

   (a) Compute the Cholesky factorization of A.

   (b) Let U be an $n \times n$ upper triangular matrix with no zeros on the diagonal, and let b be a vector in $\mathbb{R}^n$. Derive the exact number of floating point operations required to solve the triangular system of equations $Ux = b$ for x.

3. Estimate the numerical value of the integral

$$I = \int_{-1}^{1} \frac{dx}{1 + x^4}$$

   with three-point Gauss-Legendre quadrature.

4. Let p_N be the best N-th degree polynomial approximation to a real-valued function $f \in L^2[0, 1]$ in the norm

$$\|v\|_2 = \sqrt{\int_0^1 v^2 \, dx}.$$

   (a) Compute the best first-degree polynomial approximation to $f(x) = x^2$.

   (b) Let $N = 1$, and suppose that f is continuous. Prove that there is at least one point $\xi \in [0, 1]$ such that $p_1(\xi) = f(\xi)$.

   (c) Suppose $f \in C^{N+1}[0, 1]$. Derive an estimate for the uniform norm

$$\|p_N(x) - f(x)\|_\infty .$$

   You may use, without proof, the generalization of (b) to $N > 1$: there are at least N distinct points $x_i \in [0, 1]$ such that $p_N(\xi_i) = f(\xi_i)$.

5. Consider numerical solution of the initial value problem $y' = f(t, y)$, $y(0) = y_0$ with stepsize h.
A step of the *theta method* is as follows:

$$t_{n+1} = t_n + h$$

$$y_{n+1} = y_n + \theta h f(t_n, y_n) + (1 - \theta) h f(t_{n+1} y_{n+1})$$

where θ is a constant parameter in the interval $[0, 1]$.

(a) After making the minimum necessary assumptions about the differentiability of f, find the *order of accuracy* for this method as a function of θ. State your differentiability assumptions clearly. Note that your answer will depend on θ.

(b) Clearly s tate the definitions of absolute stability and A-stability. For what (if any) values of θ is this method A-stable?

6. Consider the fixed point iteration $x_{k+1} = g(x_k)$, where

$$g(x) = \tan^{-1}(2x).$$

(a) Clearly $g(x)$ has a fixed point at $x = 0$. Show that the fixed point iteration will not converge to this fixed point.

(b) There is another fixed point x_* near $x = 1.166$. Find an appropriate interval about x_* and prove that the fixed point iteration will converge to x_* for any initial guess x_0 in this interval. [Hint: $\tan^{-1}(2) \approx 1.107$.]

(c) Write out a step of Newton's method for finding x_*. (You do not need to show convergence for Newton's method.)