

Numerical Analysis Preliminary Examination, May 2013

Work **all** six of the following problems.

- Let A be a full column rank $m \times n$ matrix with singular value decomposition (SVD) $A = U\Sigma V^*$, where V^* indicates the conjugate transpose of V .
 - Compute the SVD of $A(A^*A)^{-1}A^*$ in terms of U , Σ , and V .
 - Describe the class of matrices for which the SVD exists.
 - Describe the properties of U , Σ , and V for the full and the reduced SVD.
 - Let $\|\cdot\| = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$ be the matrix norm induced by the vector 2-norm, and let σ_{\max} be the largest singular value of A . Show that $\|A\| = \sigma_{\max}$.
- Let A be a real $n \times n$ symmetric positive definite (SPD) matrix, and suppose that A is sparse with at most k nonzero entries per row. Let b be a vector in \mathbb{R}^n . We wish to solve a linear system $Ax = b$ for x .
 - What operation is the dominant cost in floating point operations in one iteration of the unpreconditioned Conjugate Gradient (CG) algorithm?
 - If we can solve the given linear system in m iterations with the CG algorithm, what is the asymptotic cost in floating point operations of solving for x ?
 - Assuming that we only have access to a naive Cholesky factorization that does not take advantage of sparsity, how does the asymptotic cost of CG for this problem compare with the cost of solving the system via Cholesky factorization?
 - If A is sparse as described, but has no other special structure, will the factor R from the Cholesky factorization $A = R^T R$ have approximately the same number of nonzero entries as the upper triangular part of A ? Explain your answer.
- Consider approximation of the integral

$$I(f) = \int_0^1 \rho(x) f(x) dx$$

by quadrature, where $\rho(x) = -\ln(x)$. In this problem you may need to use definite integral $\int_0^1 x^p \ln(x) dx = -\frac{1}{(p+1)^2}$ for $p > -1$.

- Explain why a closed Newton-Cotes rule is not an appropriate choice for $I(f)$.
- Suppose the node $x_1 = \frac{1}{2}$ is used to construct a one-point quadrature rule for computing $I(f)$.
 - What weight w_1 should be used?
 - What is the largest space of polynomials that can be integrated exactly by this rule?
- Find the one-point Gaussian rule for $I(f)$. What is the largest space of polynomials that can be integrated exactly by this rule?

4. Consider the nonlinear equation $x + \ln x = 0$ and the following possible iterations for finding its root:

$$\{i\} \quad x_{k+1} = -\ln x_k \qquad \{ii\} \quad x_{k+1} = e^{-x_k}$$

- (a) Show that one of these iterations will converge to the root and that the other will not. (Hint: you may want to sketch a graph of the function to get a rough estimate of the value of the root.)
- (b) Write out another iterative method for finding the root that has a faster rate of convergence than the convergent method from (a).
5. Let P^N be the space of polynomials of degree less than or equal to N . Given a function $f \in C^0[-1, 1]$ the *moment problem* is to find $p_N \in P^N$ such that

$$\int_{-1}^1 x^k f(x) dx = \int_{-1}^1 x^k p_N(x) dx$$

for $k = 0, 1, 2, \dots, N$.

- (a) Prove that the moment problem has a unique solution.
- (b) Given $f(x) = x^3$ and $N = 1$, solve the moment problem for p_1 .
- (c) The solution p_N to the moment problem for f can be regarded as a polynomial approximation to f . What can you say about the quality of this approximation? Is this approximation a “best” approximation? Justify your answer.
6. Consider numerical solution of the initial value problem $y' = f(x, y)$, $y(0) = y_0$ with stepsize h .
- (a) Define the following:
- i. Local truncation error
 - ii. Global truncation error
 - iii. Absolute stability
 - iv. A-stability
- (b) A step of the *midpoint method* is as follows:

$$\begin{aligned} x_{n+1} &= x_n + h \\ K_1 &= f(x_n, y_n) \\ K_2 &= f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_1\right) \\ y_{n+1} &= y_n + hK_2. \end{aligned}$$

- i. After making the minimum necessary assumptions on the differentiability of f , find the local truncation error for this step. Clearly state the necessary differentiability assumptions on f .
- ii. Find the maximum timestep for which this method is absolutely stable.