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Numerical Analysis Preliminary Examination
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August 2014

- Do all 9 problems.

1. Let $\Omega = [a, b]$ be compact, and let P^N be the space of polynomials of degree $\leq N$. Suppose p_N is the best L^2 approximation to $f \in C^{(N+1)}(\Omega)$ from P^N. Prove or give a counterexample: there exist at least N points $\zeta \in (a, b)$ at which $f'(\zeta) = p'_N(\zeta)$.

2. Let $[a, b]$ be compact, and suppose $Q_N(f)$ be the $N+1$-point Gaussian quadrature rule for the approximation of
$$I(f) = \int_a^b \rho(x) f(x)\, dx$$
where $\rho(x) > 0$ on (a, b).
   (a) Prove that the weights of Q_N are positive.
   (b) Suppose f is continuous on $[a, b]$. Prove that $Q_N(f) \to I(f)$ as $N \to \infty$.

3. Suppose a matrix A has distinct eigenvalues.
   (a) Describe the power method for computing the dominant eigenpair of A.
   (b) Explain how to adapt the power method to find the eigenpair with eigenvalue nearest to some specified complex number σ.

4. Suppose $a > 0$. One might think to compute $\sqrt{a}$ by solving $x^2 = a$ with fixed-point iteration,
$$x_{n+1} = \frac{a}{x_n}.$$
   (a) Show that this iteration does not converge unless $x_0 = \sqrt{a}$.
   (b) Propose a better algorithm for computing $\sqrt{a}$, and prove that there is an open ball $B \ni \sqrt{a}$ such that the algorithm converges to $\sqrt{a}$ for all initial guesses $x_0 \in B$.

5. Prove that a real matrix A has a Cholesky factorization iff A is symmetric positive definite.

6. Let A be an $m \times n$ real matrix with $m \geq n$, and let b be a vector in $\mathbb{R}^m$. Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^m$, and let the function $f : \mathbb{R}^n \to \mathbb{R}$ be defined as the squared residual
$$f(x) = \|Ax - b\|^2.$$
Prove that f has a unique minimum at the point x^*, where x^* is the solution to the normal equations $A^T A x^* = A^T b$.

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7. Let A be $m \times n$, with $m \geq n$ and full column rank. The *pseudoinverse* of A is denoted by A^+ and is defined by

$$A^+ = (A^* A)^{-1} A^*.$$

Show how to use the SVD to compute A^+. Why is the requirement of full column rank necessary?

8. The backward Euler method approximates the solution of

$$y' = f(x, y)$$

using the step formula

$$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1})$$
$$x_{n+1} = x_n + h.$$

(a) Find the local truncation error of this method, given suitable differentiability assumptions.

(b) Is this method A-stable? Justify your answer.

9. Let A be an $m \times m$ square matrix and $\|\cdot\|$ be some matrix norm induced by a vector norm.

(a) Prove that $\|I - A\| < 1$ implies that A is nonsingular.

(b) Prove that if $\|A\| < 1$, then

$$\sum_{k=0}^{N} A^k \to (I - A)^{-1}$$

as $N \to \infty$ in every matrix norm.