Numerical Analysis Preliminary Examination
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Do all nine problems.

1. Let $\Omega = [a, b]$ be a compact interval in $\mathbb{R}$, and let P^N be the space of real polynomials of degree at most N. Suppose p_N is the best approximation to $f \in C^{(N+1)}(\Omega)$ from P^N in the L^2 norm defined by
$$\|v\|_2 = \sqrt{\int_a^b v(x)^2\, dx}.$$
Find an upper bound on $\|p_N - f\|_2$.

2. The Hermite polynomials obey the recurrence relation
$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$
and are orthogonal with respect to the inner product
$$(u, v) = \int_{-\infty}^{\infty} e^{-x^2} u(x)\, v(x)\, dx.$$
   (a) Find the nodes and weights for the two-point Gaussian quadrature rule for approximation of
$$I(f) = \int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx.$$
You may need the formula $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.
   (b) For what space of functions is this two-point rule exact?

3. Let A be an $n \times n$ real symmetric positive definite matrix. Prove that the minimum value of the Rayleigh quotient
$$R(x) = \frac{x^T A x}{x^T x}$$
over all $x \in \mathbb{R}^n \backslash 0$ is equal to the minimum eigenvalue of A. If you like, you may make the simplifying assumption that A has distinct eigenvalues.

4. Prove that the equation
$$x = \cos(x)$$
has a unique solution, and that fixed point iteration
$$x_{n+1} = \cos(x_n)$$
converges to that solution starting from any initial guess $x_0 \in \mathbb{R}$.

5. Find the Cholesky factorization of

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 5 \\ 3 & 5 & 11 \end{pmatrix}.$$

6. Let A be an $m \times m$ real symmetric positive definite matrix, and let b be a vector in $\mathbb{R}^m$. Let x_* be the solution to $Ax_* = b$, and let the function $f : \mathbb{R}^m \to \mathbb{R}$ be defined as

$$f(x) = \frac{1}{2} x^T A x - x^T b.$$

Prove that f has a unique minimum at x_*.

7. Let A be a matrix of size $m \times n$. Prove that $\|A\|_2$ is equal to the largest singular value of A.

8. The midpoint method approximates the solution of the initial value problem

$$y' = f(x, y)$$

$$y(x_0) = y_0$$

using the step formula

$$\tilde{y}_{n+\frac{1}{2}} = y_n + \frac{h}{2} f(x_n, y_n)$$

$$x_{n+\frac{1}{2}} = x_n + \frac{h}{2}$$

$$y_{n+1} = y_n + hf\left(x_{n+\frac{1}{2}}, \tilde{y}_{n+\frac{1}{2}}\right)$$

$$x_{n+1} = x_n + h.$$

(a) Find the local truncation error of this method. State whatever differentiability assumptions are needed.

(b) Determine whether this method is A-stable.

9. An $m \times m$ matrix A is called strictly row diagonally dominant if, for each $i = 1, 2, \cdots, m$, the elements in row i obey the inequality

$$|A_{ii}| > \sum_{j \neq i} |A_{ij}|.$$

(a) Prove that strict row diagonal dominance of A implies that A is nonsingular.

(b) Let D be the diagonal part of A. Jacobi's iteration for solving $Ax = b$ is

$$x^{(n+1)} = D^{-1} b - D^{-1} (A - D) x^{(n)}$$

starting from an initial guess $x^{(0)}$. Prove that if A is strictly row diagonally dominant, then Jacobi's iteration converges (in every matrix norm) to the unique solution of $Ax = b$.