Numerical Analysis Preliminary Examination, May 2015

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Solve 8 out of 9 (eight out of nine) of the following problems. Clearly indicate which eight are to be graded.

1. Determine, if it exists, a quadrature rule of the form

$$\int_0^1 f(x)dx \approx \alpha(f(x_0) + f(x_1))$$

that is exact for all quadratic polynomials.

2. (a) Given any vector norm $\|\cdot\|_v$ in $\mathbb{R}^N$, give the definition of subordinate (or induced) matrix norm $\|\cdot\|_m$ to the given vector norm $\|\cdot\|_v$.

   (b) Let A be a square matrix. Prove that, if $\|A\|_m < 1$, then $(I - A)$ is invertible, and

$$(I - A)^{-1} = \sum_{i=0}^{\infty} A^i .$$

3. Prove that the eigenvalues of a square matrix A lie in the intersection of the two sets D and E defined by

$$D = \bigcup_{i=1}^{n} \left\{ z \in \mathbb{C} : |z - a_{ii}| \le \sum_{j=1, j\neq i}^{n} |a_{ij}| \right\} , \quad E = \bigcup_{i=1}^{n} \left\{ z \in \mathbb{C} : |z - a_{ii}| \le \sum_{j=1, j\neq i}^{n} |a_{ji}| \right\} .$$

4. Given any $r > 0$, Heron's algorithm for computing $\sqrt{r}$ is

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right) .$$

   (a) Show that $\sqrt{r}$ is a fixed point of this iteration.

   (b) Show that this iteration can be derived as a Newton method.

   (c) Prove that the sequence x_n converges to $\sqrt{r}$, where the starting point belongs to an interval to be chosen by you as needed.

5. Given a linear space E endowed with an inner product $< \cdot, \cdot >$ and corresponding norm $\|\cdot\| = \sqrt{< \cdot, \cdot >}$, prove Schwarz's inequality

$$| < f, g > | \le \|f\| \|g\| \quad \forall f, g \in E .$$

6. Determine the best approximation of the function $f(x) = x^7 + 1$ in the linear space $G = \mathrm{span}\{x, x^3, x^5\}$ on the interval $[-1, 1]$, using the norm

$$\|f\|_2 = \left(\int_{-1}^{1} |f(x)|^2 \mathrm{d}x \right)^{1/2} .$$

7. For the numerical solution of the Initial Value Problem

$$x' = f(t, x)\,, \quad x(t_0) = x_0$$

consider the implicit trapezoid method

$$x_n - x_{n-1} = \frac{1}{2} h (f_n + f_{n-1})\,,$$

where $f_n = f(t_n, x_n)$.

 (a) Show that the method is the one-step Adams-Moulton (i.e. implicit) formula and that it is equivalent to the integration of the original equation by the trapezoid quadrature rule.
 (b) Determine whether this method is convergent.
 (c) Find the order of the local and global truncation errors.
 (d) Determine whether the method is A-stable.

8. Consider the finite-difference solution of the linear Boundary Value Problem for the function $x(t)$ on the interval $[a, b]$,

$$\begin{cases} x'' = u(t) + v(t)x + w(t)x' \\ x(a) = \alpha \\ x(b) = \beta\,. \end{cases}$$

Assume $v(t) > 0$, and divide the interval $[a, b]$ using equally spaced nodes.

 (a) Propose finite-difference approximations for x' and x''.
 (b) Derive the linear system resulting from those approximations, and discuss the conditions for which the matrix is nonsingular.

9. Consider an orthonormal system $[u_1, u_2, \ldots]$ in an inner product space E. Let $U_n = \mathrm{span}\{u_1, u_2, \ldots, u_n\}$. Define the operator $P_n : E \to U_n$ by

$$P_n f = \sum_{i=1}^{n} < f, u_i > u_i\,, \qquad f \in E\,.$$

Show that

 (a) P_n is onto (surjective);
 (b) $P_n f$ is the best approximation of f in U_n.