Numerical Analysis Preliminary Exam
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1. Determine whether the matrix
$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 5 \\ 3 & 5 & 11 \end{bmatrix}$$
is symmetric positive definite.

2. Let $h(x)$ be the two-node cubic Hermite approximation to $f(x) = \cos\left(\frac{\pi}{2}x\right)$ based on the nodes $\{0, 1\}$.

   (a) Find $h(x)$.

   (b) Derive an upper bound on the error $|f(x) - h(x)|$.

3. Let $A \in \mathbb{C}^{n\times n}$ be a nonsingular matrix.

   (a) Write out the steps of the "pure" unshifted QR algorithm for finding the eigenvalues of A.

   (b) Show that each of the matrices A_k from the QR algorithm is unitarily similar to A.

   (c) Derive the asymptotic number of flops needed for a single QR factorization and use this number to approximate the number of flops required to compute the eigenvalues of A with the QR algorithm assuming the method takes k iterations to converge.

4. Let $A \in \mathbb{R}^{m\times n}$ have an SVD $A = U\Sigma V^T$.

   (a) Find eigenvalue decompositions of A^TA and AA^T in terms of the singular values and singular vectors of A.

   (b) Assuming A is square, $m = n$, find an eigenvalue decomposition of
$$S = \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}.$$
   Write the eigenvalues of S in terms of the singular values of A, and write the eigenvectors of S in terms of the right and left singular vectors of A.

5. Consider a one-point quadrature rule $Q_1(f)$ for the approximation of
$$I(f) = \int_0^\infty x^2 e^{-x} f(x)\, dx.$$

   (a) Find a node x_1 and weight w_1 such that Q_1 is exact for all polynomials of degree ≤ 1.

   (b) Derive an upper bound on the error $E(f) = |I(f) - Q_1(f)|$, stating differentiability conditions on f as needed.

   Point of information: $\int_0^\infty x^n e^{-x}\, dx = n!$.

6. Use two iterations of Newton's method to estimate $\sqrt[3]{2}$ starting from the initial guess $x_0 = 1$. Do all work in exact rational arithmetic.

7. Consider the central difference formula for the first derivative,
$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}.$$
   Derive estimates for truncation error and roundoff error, and find the stepsize h that minimizes the combined truncation and roundoff error.

8. Let $f \in C[-1,1]$. Consider the problem of finding the least squares fit to $f(x)$ by a polynomial of degree less than or equal to n. That is, find $p^*(x)$ such that

$$\int_{-1}^{1}(f(x)-p^*(x))^2dx \leq \int_{-1}^{1}(f(x)-p(x))^2dx$$

for all polynomials p of degree less than or equal to n.

(a) If $p^*(x) = a_0 + a_1x + \cdots + a_nx^n$, show how to determine the values of the coefficients $a_0, a_1, \ldots, a_n$ by solving a system of linear equations.

(b) The Legendre polynomials, $P_i(x),\ i = 0, 1, \ldots, n$, are orthogonal on $[-1,1]$ with respect to the weight function $w(x) = 1$. Show how to express $p^*(x)$ in terms of these polynomials.

Do **one** of problems 9 and 10. If you do work on both problems, clearly indicate which problem you want to be graded. If no unambiguous indication is made, problem 9 will be graded.

9. Consider a two-stage explicit Runge-Kutta (ERK) method for solving the initial value problem $y' = f(t,y)$.

   (a) Derive the order conditions for a two-stage second-order ERK method.

   (b) Find any solution to the order conditions (thereby finding a specific 2-stage ERK method of order 2).

   (c) Find the largest $h \in \mathbb{R}$ such that the method found in (b) produces a decaying solution to $y' = -1000y$, $y(0) = 1$.

   (d) Suggest a second-order method for which the choice of timestep h is not limited by stability concerns.

10. Consider the multistep method for the autonomous initial-value problem $y'(t) = f(y(t))$ of the form

$$y_n + (A-1)y_{n-1} - A\,y_{n-2} = \frac{h}{12}\left((5-A)f_n + 8(1+A)f_{n-1} + (5A-1)f_{n-2}\right),$$

where $f_n := f(y_n)$. For any $A \in \mathbb{R}$, this formula is exact for all cubic polynomial solutions $y(t) = a + bt + ct^2 + dt^3$. That is,

$$y(t_n) + (A-1)y(t_{n-1}) - A\,y(t_{n-2}) =$$
$$\frac{h}{12}\left((5-A)y'(t_n) + 8(1+A)y'(t_{n-1}) + (5A-1)y'(t_{n-2})\right)$$

for all cubic polynomials where $t_n = nh$ and h is the step length.

   (a) There exists a unique value of A so that the method is exact for all polynomials of degree 4. Determine that value.

   (b) Analyze consistency and zero-stability of the multistep method using the value of A found in part (a).

Do **one** of problems 11 and 12. If you do work on both problems, clearly indicate which problem you want to be graded. If no unambiguous indication is made, problem 11 will be graded.

11. Consider the boundary-value problem

$$y''(x) + y'(x) + xy(x) = \cos(x),\ 0 < x < 1,$$

with $y(0) = 1$, $y(1) = 2$.

   (a) Show how the solutions of two initial-value problems

$$u''(x) + u'(x) + xu(x) = \cos(x),\ 0 < x < 1, u(0) = 1, u'(0) = 0$$
$$w''(x) + w'(x) + xw(x) = \cos(x),\ 0 < x < 1, w(0) = 1, w'(0) = 1$$

   can be combined to find $y(x)$. In particular, find a and b such that $y(x) = au(x) + bw(x)$.

   (b) Describe a numerical procedure for approximating $y(x)$ using part (a).

12. Consider the boundary-value problem

$$u''(x) = f(x)$$
$$u(0) = u(1) = 0.$$

Define a function space in which a weak solution can be found, and derive the Galerkin weak form of this problem.