Numerical Analysis Preliminary Exam
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1. Find all $\beta \in \mathbb{R}$ for which the matrix
$$A = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 17 & 3 \\ 6 & 3 & \beta \end{bmatrix}$$
is symmetric positive definite.

2. Let $A \in \mathbb{R}^{n \times n}$ have n distinct eigenvalues $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n| > 0$ and associated eigenvectors $v_1, v_2, \ldots, v_n$.
   (a) Describe the power method for computing the dominant eigenpair of A.
   (b) Given certain conditions on the initial guess $v_1^{(0)}$ (which you must specify), prove that the power method converges to the eigenpair λ_1, v_1 when calculations are done in exact arithmetic.
   (c) Estimate the flop count for finding λ_1, v_1 with the power method assuming k iterations are needed for convergence.
   (d) Explain how to adapt the power method to find the eigenpair with eigenvalue nearest some specified number μ.

3. Suppose A is an $m \times n$ real matrix. Let $\|\cdot\|_2$ be the matrix 2-norm and let $\sigma_{\max}$ be the largest singular value of A.
   (a) Show that $\|A\|_2 = \sigma_{\max}$.
   (b) Let α be any positive real number. Find the singular value decomposition of $\alpha I + A^T A$.
   (c) Use the result of part (b) to solve the linear system $(\alpha I + A^T A)\, x = A^T b$

4. Let $p_2(x)$ be the quadratic Lagrange interpolating polynomial approximation to $f(x) = e^x$, based on the nodes $\left\{0, \frac{1}{2}, 1\right\}$. Derive an upper bound on the error $|f(x) - p_2(x)|$.

5. Let $Q_3(f)$ be the three-point Gauss-Legendre quadrature rule for the approximation of
$$I(f) = \int_{-1}^{1} f(x)\, dx.$$
   (a) Find the nodes and weights for Q_3
   (b) Prove that Q_3 is exact for all $f \in \mathbb{P}^5$.

6. Prove that the iteration
$$x_{n+1} = \frac{1}{2} \cos(x_n)$$
converges to a unique fixed point when started from any point $x_0 \in \mathbb{R}$.

7. Referring to the properties of floating point arithmetic and the concept of condition number, explain why $\sin\left(10^{100}\right)$ cannot be computed to any useful accuracy in standard double-precision floating point arithmetic.

8. Let $\{\phi_1, \phi_2, \ldots, \phi_n\}$ be an orthonormal set in an inner-product space V with the associated inner-product $(\cdot, \cdot)$. Let V^n be the subspace generated by $\phi_1, \phi_2, \ldots, \phi_n$. Let $f \in V$ and $g^* \in V^n$ satisfy $f - g^* \perp V^n$.
   (a) Show that g^* is the best approximation to f from V^n. That is, show that $\|f - g^*\| \le \|f - g\|$ for any $g \in V^n$. Also show that $g^* = \sum_{i=1}^{n} (f, \phi_i)\, \phi_i$.
   (b) Let the norm $\|\cdot\|$ be defined as $\|f\| = \sqrt{(f, f)}$. Prove that $\sum_{i=1}^{n} |(f, \phi_i)|^2 \le \|f\|^2$.

Do **one** of problems 9 and 10. If you do work on both problems, clearly indicate which problem you want to be graded. If no unambiguous indication is made, then problem 9will be graded.

9. A step of the trapezoidal rule for solving the initial value problem $y' = f(t, y)$, $y(0) = y_0$ is given by the formula

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_n + h, y_{n+1})\right].$$

It can be shown that this is a second order method and is A-stable (you do **not** need to prove either of those facts here). The practical difficulty is that because y_{n+1} appears both on the LHS of the equation and also on the RHS as an argument to f, an algebraic equation must be solved for y_{n+1} at each step. It is therefore often useful to replace y_{n+1} on the RHS by an approximation, avoiding the need to solve an implicit equation.

   (a) Show that approximating y_{n+1} on the RHS by $\tilde{y} = y_n + hf(t_n, y_n)$ produces a method that is also second order.

   (b) Show that the method devised in part (a) is *not* A-stable.

10. For the numerical solution of the Initial Value Problem

$$y' = f(t, y), \quad y(t_0) = y_0$$

consider the multistep method

$$y_n + \alpha y_{n-1} - (1+\alpha)y_{n-2} = \frac{1}{2}h[-\alpha f_n + (4 + 3\alpha)f_{n-1}]$$

where $f_n = f(t_n, y_n)$.
Determine α so that the method is consistent, zero-stable, convergent, A-stable and of second order.

Do **one** of problems 11 and 12. If you do work on both problems, clearly indicate which problem you want to be graded. If no unambiguous indication is made, then problem 11 will be graded.

11. Consider the two-point boundary-value problem

$$y''(x) - p(x)y'(x) - q(x)y(x) = r(x),\ 0 < x < 1,\ y(0) = y(1) = 0.$$

Assume that $q(x) \geq \alpha > 0$ for $0 \leq x \leq 1$. Consider the difference scheme

$$\frac{(y_{j+1} - 2y_j + y_{j-1})}{h^2} - p(x_j)\frac{(y_{j+1} - y_{j-1})}{2h} - q(x_j)y_j = r(x_j),$$

for $j = 1, 2, \cdots, N-1$, with $y_0 = y_N = 0$, $x_j = jh$, and $h = 1/N$.

   (a) Determine the matrix A so that the above difference equations can be written as the linear system $Ay = h^2 r$ with $y = [y_1, y_2, \cdots, y_{N-1}]^T$ and $r = [r(x_1), r(x_2), \cdots, r(x_{N-1})]^T$.

   (b) Prove that if

$$\frac{h}{2}\max_{0\leq x\leq 1}|p(x)| \leq 1,$$

   then the $(N-1) \times (N-1)$ matrix A is strictly diagonally dominant.

12. Let H_0^1 be the function space

$$H_0^1 = \left\{v : v(0) = 0, v(1) = 0, \int_0^1 \left(\frac{dv}{dx}\right)^2 dx < \infty\right\}.$$

Consider the weak problem

$$\int_0^1 v'u' + ve^x\, dx = 0 \ \ \forall v \in H_0^1.$$

Derive the 2-point boundary value problem corresponding to this weak problem.